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Marches aléatoires avec branchement et absorption

Résumé

Nous étudions des marches aléatoires branchantes unidimensionnelles, où une barrière absorbante tue avant qu'ils ne se reproduisent les individus qui la franchissent. Par des méthodes probabilistes, nous obtenons des résultats qui fournissent des informations sur les trajectoires des marches branchantes classiques (sans barrière absorbante).

Dans le cas où la barrière est fixée à l'origine, nous estimons la vitesse d'extinction dans les cas critique et sous-critique.

Nous affinons ensuite l'étude du cas critique en considérant une barrière du second ordre dont la position est proportionnelle à la puissance 1/3 de la génération. Nous déterminons la valeur limite de la position de la barrière séparant la survie avec probabilité positive de l'extinction presque sûre. Dans le cas d'extinction nous évaluons la probabilité de survie et la queue de distribution de la population totale.

Dans la dernière partie de cette thèse, nous nous plaçons dans un cadre un peu différent où la position de la barrière dépend du nombre de générations considérées. Nous obtenons un résultat de déviations modérées sur le déplacement minimal de la marche branchante qui fait apparaître différents régimes, en fonction de la queue de distribution de la loi des déplacements.

Mots-clefs : Marche aléatoire branchante, absorption, probabilité de survie.

Branching random walk with absorption

Abstract

We study unidimensional branching random walk with an absorbing barrier that kills the individuals that cross it before they reproduce. We use probabilistic methods and obtain results that provide information on the trajectories of branching random walks without absorption.

In the case of a barrier set at the origin, we estimate the extinction rate in the critical and subcritical cases.

Then we refine the study of the critical case by considering a second order barrier. We determine the relevant order of the position of this barrier, which is proportional to the generation to the power 1/3, and the boundary position of the barrier which separates the survival and the extinction cases. In the extinction case, we also estimate the survival probability and the tail of the distribution of the total progeny.

The setup of the last part of this thesis is a bit different since the position of the barrier depends on the number of generation we consider. We obtain a moderate deviation result for the consistent minimal displacement of the branching random walk, which presents various regimes depending on the tail of distribution of the law of the displacements.

Keywords : Branching random walk, absorption, survival probability.

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Chapitre 1

Introduction

1 Les marches aléatoires branchantes

Les marches aléatoires branchantes généralisent simultanément deux modèles à temps discret très classiques de la théorie des probabilités : le processus de Galton-Watson et la marche aléatoire. Nous nous intéressons uniquement aux marches aléatoires branchantes unidimensionnelles mais on pourrait bien sûr en définir de la même manière en dimension quelconque ou sur des graphes généraux.

Le processus est initialisé au temps $n = 0$ avec un unique individu placé en $x \in \mathbb{R}$. Au temps $n = 1$, cet individu meurt et donne naissance à une première génération d'individus dont le nombre N et les déplacements $\xi(i) \in \mathbb{R}$, $1 \leq i \leq N$ par rapport à leur parent sont donnés par un processus ponctuel $Z = \{\xi(1), \ldots \xi(N)\}$ (avec N aléatoire). Ainsi le $i^{\text{ème}}$ individu de la première génération a pour position $x + \xi(i)$. Au temps $n = 2$, chaque individu de la première génération meurt et donne naissance à des enfants, dont le nombre et les déplacements sont donnés par des copies i.i.d. de Z. On obtient ainsi la seconde génération, et le processus continue indéfiniment (à moins qu'il y ait extinction), chaque individu d'une génération se reproduisant indépendamment des autres et indépendamment du passé. On note ξ_u le déplacement de l'individu u par rapport à son parent. Le déplacement total de l'individu u par rapport à la racine est donc la somme des déplacements de ses ancêtres (u inclus mais pas la racine, dont le déplacement n'est pas défini), et la position de u est donc :

$$V(u) := x + \sum_{v \leq u} \xi_v.$$

De manière évidente, la population forme un arbre de Galton-Watson $\mathcal{T}$ et celui-ci est enrichi par la donnée de la position des individus. Dans le cas le plus élémentaire, Z est tel que les déplacements des individus de la première génération sont i.i.d. et indépendants de leur nombre. Alors, pour tout individu $u \in \mathcal{T}$, en notant $|u|$ sa génération et u_i ($i \leq |u|$) son ancêtre dans la génération i, le processus $(V(u_i))_{0\leq i\leq |u|}$ est une marche aléatoire issue de x. Lorsqu'on choisit deux individus différents de la $n^{\text{ème}}$ génération, on obtient deux marches qui coïncident jusqu'au plus proche ancêtre commun puis se comportent de manière indépendante ensuite.

Nous nous intéressons exclusivement au cas où le processus de Galton-Watson sous-jacent est surcritique. Alors, avec probabilité strictement positive le processus survit indéfiniment (et même avec probabilité 1 si on suppose que chaque individu a au moins un enfant). La population croît alors à vitesse exponentielle. Une question naturelle est le comportement des particules occupant des positions extrémales, par exemple la position R_n de l'individu le plus à droite de la génération n (pour le plus à gauche, cela revient à considérer le plus à droite dans le processus obtenu en changeant Z en $-Z$).

Avant d'énoncer des résultats, il est utile d'introduire la mesure intensité μ de Z et sa fonction génératrice des moments :

$$\Phi(t) := \int_{\mathbb{R}} \mathrm{e}^{tz}\mu(\mathrm{d}z) = \mathbb{E}\left[\sum_{|u|=1} \mathrm{e}^{t\xi_u}\right].$$

On définit aussi $\Psi = \log \Phi$, c'est une fonction convexe et semi-continue inférieurement à valeur dans $(-\infty, +\infty]$. On suppose que le nombre moyen d'enfants $\Phi(0) = \mu(\mathbb{R})$ est fini.

La théorie des marches aléatoires branchantes repose sur l'hypothèse que $\Phi(t)$ est fini pour un certain t strictement positif. Cette hypothèse n'est pas un simple outil technique, elle est en fait essentielle. Si les déplacements ont une queue de distribution à droite trop lourde, par exemple polynomiale, alors le comportement est sensiblement différent. Par exemple, Durrett [37] a montré que dans ce cas (et en supposant $\mu((0, +\infty)) > 0$ et une queue de ditribution à gauche "raisonnable"), R_n croît exponentiellement (au lieu de linéairement lorsque les déplacements admettent des moments exponentiels).

On introduit la fonction de partition

$$A_n(t) := \mathbb{E}\left[\sum_{|u|=n} \mathrm{e}^{tV(u)}\right].$$

Pour tout t tel que $\Phi(t) < +\infty$, $W_n(t) = A_n(t)\Phi(t)^{-n}$ est une martingale, dite martingale additive.

2 Historique

Théorème 2.A (Hammersley-Kingman-Biggins). *Sous l'hypothèse que $\Phi(t) < +\infty$ pour un certain $t > 0$, conditionnellement à la survie, R_n/n converge p.s. lorsque n tend vers l'infini, vers une limite v_c, la vitesse de la marche branchante, caractérisée par :*

$$v_c := \inf_{t>0} \frac{\Psi(t)}{t} = \sup\{a \in \mathbb{R} : \sup_{t\geq 0} ta - \Psi(t) < 0\}.$$

Ce résultat a d'abord été démontré pour des marches à déplacements tous négatifs. Plus précisément, en retournant l'espace des positions, Hammersley [47] a démontré pour la convergence en probabilité pour la vitesse de l'individu le plus à gauche pour un processus de Bellman-Harris. Kingman [65] a ensuite démontré la convergence presque sûre pour un processus de Crump-Mode-Jagers en supposant que pour un certain $t > 0$, $1 < \Phi(t) < +\infty$. Biggins [9] a ensuite supprimé cette restriction et étendu le résultat aux marches aléatoires branchantes.

L'analogue continu de ce modèle, le mouvement brownien branchant (avec branchement binaire), a aussi fait l'objet d'études poussées. L'utilisation de méthodes spécifiques au cas continu comme le calcul stochastique, et en particulier l'exploitation du lien (voir McKean [89]) avec l'équation aux dérivées partielles F-KPP (introduite en 1937 par Fischer [42], Kolmogorov, Petrovski et Piscounov [66]) dont les solutions convergent vers une onde progressive, a permis d'obtenir des résultats très fins sur la position de la particule la plus à droite. Bramson [20], [22] a montré la tension puis la convergence en distribution de $R_t - m_t$, où $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O(1)$ est la médiane de R_n (en particulier $v_c = \sqrt{2}$).

Des résultats similaires ont été obtenus plus récemment pour les marches aléatoires branchantes, sauf dans certains cas pathologiques. Le résultat est en effet différent quand les déplacements sont majorés et le nombre moyen de particules dont le déplacement est maximal supérieur à 1. Pour le voir, il suffit de considérer le sous-arbre formé par les individus dont le déplacement est maximal, qui forment un processus de Galton-Watson surcritique (et le comportement est alors évident) ou critique (voir par exemple Bramson [21]). Dans les "bons cas", sous différentes hypothèses techniques, McDiarmid [88] (qui suppose les déplacements majorés), Bramson et Zeitouni [19], Addario-Berry et Reed [2] ont montré la tension de la loi de la particule la plus à droite de la génération n centrée sur sa médiane et obtenu la même estimation du centrage que dans le cas continu. Bachmann [7] a démontré la convergence en distribution – qui nécessite d'écarter le cas où les déplacements sont portés par un sous groupe discret de $\mathbb{R}$ – dans le cas de déplacements i.i.d. admettant une densité log-concave.

Il faut aussi signaler le modèle des cascades multiplicatives, équivalent à celui des marches de branchement mais présenté sous un formalisme différent et étudié avec des méthodes et en vue d'objectifs autres, qui a fait l'objet d'une littérature abondante depuis son introduction par Mandelbrot pour l'étude d'un problème de turbulence issu de la physique. Kahane [56], [57], [58], [59] et Kahane et Peyrière [60] ont apporté une contribution importante à cette théorie. Mauldin et Williams [87] déterminent la dimension d'objets fractaux aléatoires, généralisant ainsi les travaux de Moran [92] dans le cas déterministe. On pourra aussi se référer dans ce domaine aux travaux plus récents de Waymire et Williams [108], [109] ou Liu [75].

Des résultats fins concernant la martingale additive et la transformation dont sa limite est un point fixe naturel ont été obtenus par Durett et Liggett [38], Liu [73], Biggins et Kyprianou [12], [14].

Parmi les méthodes développées pour l'étude des marches aléatoires branchantes, les constructions à épine méritent une attention particulière. Elles permettent, par un changement de probabilité, de choisir de manière assez naturelle un individu, de manière aléatoire mais en privilégiant ceux les plus à droite. Elles ont en particulier permis à Lyons, Pemantle et Peres [82] et Lyons [80] de redémontrer, plus simplement et avec hypothèses plus faibles, le théorème de Biggins [10] qui généralise le théorème de Kesten-Stigum en donnant une condition nécessaire et suffisante pour la dégénérescence de la limite de la martingale additive. Cet outil a été utilisé à de nombreuses reprises par la suite.

Notons que si les marches aléatoires branchantes unidimensionnelles ont un intérêt propre, on peut aussi les considérer comme le potentiel dans lequel évolue une marche aléatoire en milieu aléatoire sur un arbre (voir Lyons et Pemantle [81], Menshikov et Petritis [90]). La compréhension de ce potentiel est indispensable afin d'obtenir des résultats fins sur la marche aléatoire en milieu aléatoire (voir par exemple Hu et Shi [51], [52]).

3 Les marches aléatoires branchantes avec barrière absorbante

A partir d'ici, on suppose qu'il existe $t^* > 0$ tel que $\Psi(t^*) = t^*\Psi'(t^*)$. Cette condition est discutée en section 7 dans le chapitre 3. L'infimum de la première formule du Théorème 2.A est alors atteint en t^* et donc $v_c = \Psi'(t^*)$.

Nous appelons cas surcritique celui où $v_c > 0$, cas critique celui où $v_c = 0$ et enfin sous-critique lorsque $v_c < 0$.

On peut évidemment soustraire v_c à chaque déplacement (autrement dit changer Z en $Z - v_c$) pour se ramener à l'étude du cas critique $v_c = 0$. On peut ensuite multiplier les déplacements par t^* (changer Z en t^*Z) pour se ramener à $t^* = 1$.

Sous ces conditions de normalisation, Hu et Shi [53] ont montré un résultat précis pour la position de la particule la plus à droite :

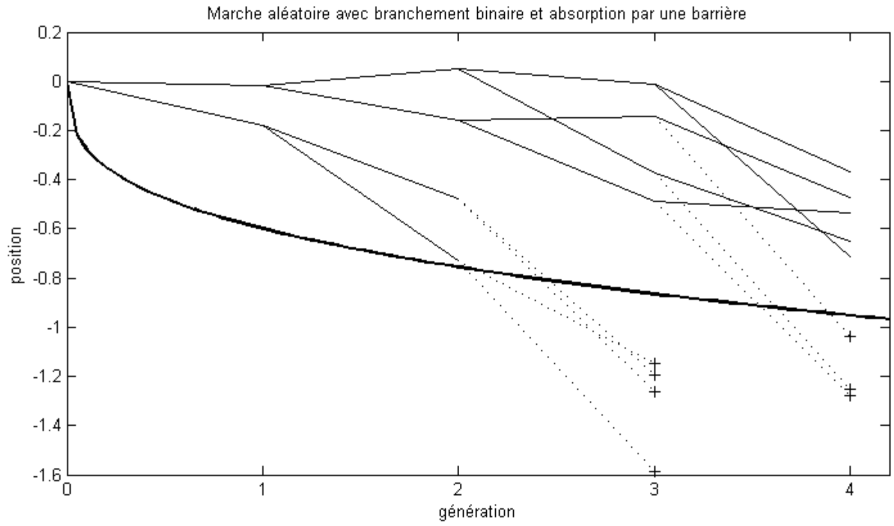
Théorème 3.A (Hu-Shi). *Conditionnellement à la survie,*

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{\log n} R_n &= -\frac{1}{2} \qquad \textit{p.s.}, \\ \liminf_{n\to\infty} \frac{1}{\log n} R_n &= -\frac{3}{2} \qquad \textit{p.s.}, \\ \lim_{n\to\infty} \frac{1}{\log n} R_n &= -\frac{3}{2} \qquad \textit{en probabilité.} \end{aligned}$$

Ainsi nous savons où se trouve la particule la plus à droite, mais ce résultat ne nous dit rien sur son historique avant le temps n. Par exemple on peut se demander si c'est l'individu le plus à droite dans une génération donnée qui engendre l'individu le plus à droite dans une génération ultérieure. La réponse, négative, à cette question particulière a été donnée par Lalley et Sellke [69] dans le cas du mouvement brownien avec branchement binaire, qui est l'analogue continu de la marche aléatoire de branchement. Leur résulat est que chaque individu a, dans une certaine génération ultérieure, un descendant qui occupe la position la plus à droite.

Dans cette thèse, nous nous intéressons aux "trajectoires extrémales" de la marche, donc à l'historique des individus et pas seulement aux positions à une génération donnée. Nous introduisons pour cela une barrière absorbante définie comme une suite $\varphi : \mathbb{N} \mapsto \mathbb{R}$, qui tue tous les individus qui la franchissent (c'est-à-dire, pour chaque $n \in \mathbb{N}$, les individus de la génération n dont la position est inférieure à $\varphi(n)$, lesquels ne se reproduisent donc pas). Nous considérons uniquement la population qui survit au dessus de la barrière.

Ces marches aléatoires branchantes avec barrières peuvent modéliser un problème d'évolution de population avec sélection : au lieu d'avoir comme dans le processus de Galton-Watson classique une même chance pour chaque individu de se reproduire, on fait dépendre son aptitude à se reproduire de son adaptation au milieu en interprétant la position d'un individu comme sa force et en considérant que seuls les individus d'une force suffisante (situés au dessus de la barrière) survivent et se reproduisent. Il y a alors compétition entre l'accroissement



exponentiel de la population dans le processus de Galton-Watson sous-jacent et l'effet de la barrière. On peut néanmoins douter de la pertinence de ce modèle dans la mesure où on ne parvient pas à obtenir de population stable : dans les cas où l'on sait distinguer la survie de l'extinction, on observe soit l'extinction du processus, soit la survie avec accroissement exponentiel de la population, comme dans le cas du processus de Galton-Watson. Nous mentionnons un autre modèle de sélection, plus fidèle à ce que l'on voudrait décrire, avec une population limitée par les ressources du milieu mais qui ne s'éteint pas nécessairement est obtenu en ne conservant que les N individus les plus performants parmi ceux obtenus en faisant se reproduire ceux de la génération précédente (voir Brunet et Derrida [23]).

La raison qui a motivé l'introduction de ce modèle par Biggins et al. [15] est en fait un problème de calculs en parallèle issu de l'informatique. On peut aussi s'intéresser aux marches branchantes avec barrières comme indiqué plus haut pour obtenir des informations sur les trajectoires d'une marche aléatoire branchante.

4 Probabilité de survie dans les cas critique et sous-critique

Nous étudions ici le cas d'une barrière linéaire se déplaçant à une vitesse v : $\forall n \in \mathbb{N}$, $\varphi(n) = nv$, et nous supposons que la position intiale x (sous $\mathbb{P}^x$) est strictement positive. Cela équivaut à considérer une barrière fixe à l'origine et à étudier la marche branchante de vitesse $\varepsilon = v_c - v$ dont les déplacements sont tous diminués de v. On définit Φ et Ψ pour cette nouvelle marche et on introduit :

$$h := \inf_{t>0} \Psi(t).$$

Notons Z_n le nombre d'individus vivants dans la génération n.

Biggins et al. [15] ont montré le résultat suivant :

Théorème 4.A (Biggins-Lubachevsky-Shwartz-Weiss). *On suppose que les déplacements sont des copies i.i.d d'une variable aléatoire réelle ξ telle que $\Phi'(0) = \Phi(0)\mathbb{E}[\xi] < 0$ et $\mathbb{P}(\xi > 0) > 0$ et que $\Phi(t) < +\infty$ pour un certain $t > 0$ (en particulier le nombre moyen d'enfants $\Phi(0)$ est fini). Alors l'infimum définissant h est atteint pour une unique valeur $\nu > 0$ de t et*

$$\frac{1}{n}\log \mathbb{E}^x[Z_n] \to h.$$

De plus,

$$\begin{aligned} \mathbb{P}^x\left(\lim_{n\to\infty} Z_n = 0\right) &= 1 \quad \text{si } \varepsilon \leq 0 \\ \mathbb{P}^x\left(\lim_{n\to\infty} Z_n = \infty\right) &> 0 \quad \text{si } \varepsilon > 0. \end{aligned}$$

En particulier, il y a extinction presque sûre dans le cas sous-critique $\varepsilon < 0$ et survie avec probabilité strictement positive dans le cas surcritique. Le théorème ne conclut pas dans le cas critique, mais on peut voir par un argument très simple (voir Addario-Berry et Broutin [1]) ou à la lumière du Théorème 3.A qu'il y a extinction presque sûre lorsque $\varepsilon = 0$.

Dans la première partie de ce travail, nous complétons l'analyse ci-dessus en estimant $\mathbb{P}^x(Z_n > 0)$ dans les cas critique et sous-critique. Les résultats sont connus dans le cas continu (mouvement Brownien branchant), grâce aux travaux de Derrida et Simon [31], [32], Harris et Harris [48] et Kesten [61], mais les méthodes employées souvent spécifiques au cas continu ne se transposent pas aisément au cas discret. Nous montrons le résultat suivant :

Théorème 4.1 (Aidékon-Jaffuel). *Nous supposons le nombre d'enfants déterministe et les déplacements i.i.d., vérifiant de plus les conditions suivantes :*

- *Il existe $s > 0$ tel que $\Phi(s) < \infty$.*
- *Il existe $\nu \in (0, s)$ tel que $\Phi'(\nu) = 0$ (et donc $h = \Psi(\nu)$).*
- *Le support de la loi ξ n'est pas inclus dans un sous groupe discret de $\mathbb{R}$.*

Dans le cas sous-critique, qui correspond à $h < 0$, il existe une constante $C_1 > 0$ telle que

$$\mathbb{P}^x(Z_n > 0) \sim C_1 \mathrm{e}^{\nu x} \widetilde{V}(x) \mathrm{e}^{nh} n^{-3/2}$$

où $\widetilde{V}$ est une fonction de renouvellement définie en (1.3) dans le chapitre 2. Dans le cas critique, qui correspond à $h = 0$,

$$\log(\mathbb{P}^x(Z_n > 0)) \underset{n\to\infty}{\sim} -\left(\frac{3\sigma^2\nu^2\pi^2}{2}\right)^{1/3} n^{1/3}.$$

où $\sigma^2 := \Phi''(\nu)/\Phi(\nu)$.

Le résultat du cas sous-critique s'étend sans difficulté au cas d'un processus de Galton-Watson admettant suffisamment de moments (un moment d'ordre 2 par exemple). En revanche les arguments utilisés ne sont plus valables si on ne suppose plus les déplacements i.i.d. : il est donc délicat d'étendre le résultat au cas général où les déplacements des frères et soeurs sont corrélés et peuvent dépendre du nombre d'enfants. Le cas où la loi de la marche a son support inclus dans un sous-groupe discret de $\mathbb{R}$ nécessite un traitement séparé mais très similaire pour des raisons techniques et un résultat analogue est attendu.

Pour le cas critique, on peut étendre le résultat au cas général pourvu que le processus de Galton-Watson sous-jacent admette suffisamment de moments, pour cela nous renvoyons au chapitre 3.

Nous terminons cette section en citant un résultat de Gantert, Hu et Shi [44] qui décrit comment la probabilité de survie (partant de 0) $\rho(\varepsilon)$ décroît dans le cas sous-critique lorsque la vitesse de la barrière $v = v_c - \varepsilon$ tend vers celle v_c de la marche :

Théorème 4.B (Gantert-Hu-Shi). *Sous les conditions techniques suivantes :*

- *Il existe* $s_1 > 0$ *et* $s_2 < 0$ *tels que* $\Psi(s_1) < \infty$ *et* $\Psi(s_2) < \infty$ *;*
- *Il existe* $t^* \in (0, s_1)$ *tel que* $t^*\Psi'(t^*) = \Psi(t^*)$ *;*
- *Le nombre d'enfants* N *vérifie* $\mathbb{E}\left[N\right] > 1$ *et* $\mathbb{E}\left[N^{1+\delta}\right] < +\infty$ *pour* $\delta > 0$ *assez petit;*

on a

$$\log \rho(\varepsilon) \underset{\varepsilon \to 0^+}{\sim} -\frac{\pi\sqrt{t^*\Psi''(t^*)}}{\sqrt{2\varepsilon}}.$$

5 La barrière critique pour la marche branchante avec absorption

L'objectif du chapitre 3 est d'affiner le résultat du Théorème 4.A. Dans le cas critique, il y a extinction presque sûre et on souhaite trouver une barrière du second ordre en $o(n)$ telle que la marche branchante survive. Plus généralement on souhaiterait déterminer, étant donné une barrière quelconque $\varphi : \mathbb{N} \to \mathbb{R}$, si la marche branchante absorbée par cette barrière survit ou s'éteint.

Théorème 5.1. *Dans le cas critique normalisé, dans le cas d'un nombre d'enfants borné ou sous d'autres hypothèses techniques adéquates, la marche branchante absorbée par la barrière*

$\varphi : n \mapsto \varphi(n) := an^{1/3}$ s'éteint presque sûrement si $a > -a_c$ et survit avec probabilité strictement positive si $a < -a_c$, où

$$a_c = \frac{3}{2}\left(3\pi^2\sigma^2\right)^{1/3} \text{ avec } \sigma^2 := \Phi''(1) = \mathbb{E}\left[\sum_{|u|=1} \xi_u^2 e^{\xi_u}\right] < +\infty.$$

En notant $\mathcal{T}_\infty$ la frontière de l'arbre (les chemins infinis vers le bas), on en déduit, sous les mêmes hypothèses, que

Théorème 5.2. *Dans le cas critique normalisé, conditionnellement à la survie,*

$$\sup_{u \in \mathcal{T}_\infty} \liminf_{n\to\infty} \frac{V(u_n)}{n^{1/3}} = -a_c.$$

Dans le cas $a = a_c$, la question de la survie ou de l'extinction reste ouverte. Nous ne disposons pas de condition nécessaire et suffisante sur une barrière générale pour la survie de la marche. Les techniques développées ici donnent des critères basés sur les limites inférieure et supérieure de $\varphi(n)/n^{1/3}$ qui ne couvrent pas tous les cas envisageables. En particulier ils ne sont précis que lorsque $\varphi(n)/n^{1/3}$ est presque toujours proche de sa limite inférieure et, rarement, autrement dit à des intervalles très espacés, de sa limite supérieure. A l'inverse, lorsque $\varphi(n)/n^{1/3}$ est souvent proche de la limite supérieure, la limite inférieure perd tout intérêt. Ces énoncés sont volontairement assez vagues car les oscillations de $\varphi(n)/n^{1/3}$ entre sa limite inférieure et sa limite supérieure sont difficiles à décrire. Les deux cas de figures évoqués ci-dessus ne couvrent pas toutes les situations, nous renvoyons au chapitre 3 pour plus de détails.

Dans le cas des barrières $\varphi : n \mapsto \varphi(n) := an^{1/3}$, nous pouvons donner des résultats plus précis que le théorème ci-dessus qui ne traite que la question de la survie et de l'extinction.

Dans le cas de survie $a < -a_c$, nous montrons, le long d'une suite, que la population croît quasi-exponentiellement avec probabilité strictement positive, et est de l'ordre de $\exp\left(cn^{1/3}(1+o(1))\right)$ à la génération n, avec c une constante qui dépend de a et qui a une limite strictement positive $2a_c/3$ lorsque $a \to -a_c$.

Dans le cas d'extinction, nous estimons la probabilité de survie, qui est de l'ordre de $\exp\left(-cn^{1/3}(1+o(1))\right)$, pour une constante c dépendant de a et telle que $c \to 0$ lorsque $a \to -a_c$.

De plus, puisque la population totale Z est finie presque sûrement, on peut s'intéresser à la queue de distribution de Z. Dans le cas $a = 0$ Aldous a conjecturé que $\mathbb{E}[Z] < +\infty$ mais

$\mathbb{E}[Z \log Z] = +\infty$. Ce résultat a été récemment démontré par Addario-Berry et Broutin [1] et amélioré par Aidékon [3] qui a donné la queue de distribution.

Dans le cas sous-critique (barrière linéaire à vitesse strictement positive), la queue de distribution est de l'ordre de $\mathbb{P}(Z > k) \sim k^{\kappa}$ pour un certain $\kappa < -1$ (travail en cours de rédaction de E. Aidékon, Y. Hu et O. Zindy). Dans le cas critique, avec une barrière du second ordre $\varphi : n \mapsto \varphi(n) := an^{1/3}$, nous obtenons une queue de distribution $\mathbb{P}(Z > k) = k^{(\kappa+o(1))}$ avec $\kappa = -1$ si $a \geq 0$ (cette estimation est grossière au vu des résultats cités plus haut) et $-1 < \kappa < 0$ si $-a_c < a < 0$.

6 Déviations modérées pour le déplacement minimal d'une marche aléatoire branchante

Dans la dernière partie de cette thèse, nous considérons une marche aléatoire branchante critique ($v_c = 0$) normalisée (à $t^* = \nu = 1$). Il est plus commode ici de changer le signe des positions, de sorte que les conditions de normalisation s'écrivent $\Psi(-1) = \Psi'(-1) = 0$, qu'on s'intéresse aux particules les plus à gauche et que la barrière tue les individus qui passent non plus en-dessous mais au-dessus d'elle.

Désignons par $\mathcal{T}_n$ la population de la génération n de la marche branchante, $|u|$ la génération d'un individu u et $V(u)$ sa position. Les techniques de la section précédente permettent aussi d'estimer certaines probabilités de survie de la forme :

$$\mathbb{P}\left(\exists u \in \mathcal{T}_n, \forall v \leq u, V(v) \leq \varphi^{(n)}(|v|)\right)$$

où la barrière $i \in \{1, \ldots, n\} \mapsto \varphi^{(n)}(i)$ dépend du temps n jusqu'auquel on demande à la marche de survivre (on retrouve exactement le cadre précédent si $\varphi^{(n)}(i)$ ne dépend pas de n). Il suffit que φ s'écrive sous la forme

$$\varphi^{(n)}(i) = n^{1/3} g\left(\frac{i}{n}\right)$$

avec $g : [0,1] \to \mathbb{R}$ une fonction "sympathique".

Dans le chapitre 3, on prenait $g(t) = -at^{1/3}$ qui faisait disparaître la dépendance en n. Ici, nous faisons un choix encore plus simple : nous supposons g constante, égale à $b \in \mathbb{R}$ sur $[0,1]$.

La survie sous la barrière jusqu'au temps n correspond alors à l'événement :

$$\left\{\min_{|u|=n} \overline{V}(u) \leq bn^{1/3}\right\} = \left\{\exists u \in \mathcal{T}_n, \overline{V}(u) \leq bn^{1/3}\right\} \tag{6.1}$$

où

$$\overline{V}(u) := \max_{v \leq u} V(v).$$

Fang et Zeitouni [39] ont montré le résultat suivant :

Théorème 6.A. *Dans le cas critique avec branchement déterministe et déplacements i.i.d.,*

$$\frac{\min_{|u|=n} \overline{V}(u)}{n^{1/3}} \to b_c \text{ p.s. lorsque } n \to \infty$$

avec

$$b_c := \left(\frac{3\pi^2\sigma^2}{2} \right)^{1/3}$$

avec σ^2 défini comme à la section précédente.

De plus, si $b < b_c$, alors

$$\frac{\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) \leq bn^{1/3}\right)}{n^{1/3}} \to b - b_c \text{ p.s. lorsque } n \to \infty.$$

La probabilité que la marche branchante soit moins rapide que le comportement typique, autrement dit la probabilité de l'événement (6.1) pour $b < b_c$ est donc connue et son comportement asymptotique ne dépend de la loi de la reproduction et des déplacements que via les conditions de normalisation et la valeur de σ^2, et il en va de même pour la valeur critique b_c.

Nous étudions ici la probabilité que la marche branchante soit plus rapide que le comportement typique, autrement dit de l'événement

$$\text{(6.2)} \qquad \left\{ \min_{|u|=n} \overline{V}(u) > bn^{1/3} \right\} = \left\{ \forall u \in \mathcal{T}_n, \overline{V}(u) > bn^{1/3} \right\}$$

pour $b > b_c$.

Le résultat attendu est que la probabilité de (6.2) doit tendre vers 0 plus rapidement que celle de (6.1), car la réalisation de l'événement (6.2) implique une contrainte sur tous les chemins qui partent de la racine, alors que pour (6.1) un seul chemin plus lent que le comportement typique suffit.

C'est effectivement ce qui se produit. On observe aussi un autre phénomène beaucoup moins attendu : la vitesse à laquelle la probabilité de (6.2) tend vers 0 pour $b > b_c$ dépend fortement de la queue de distribution à droite de la loi des déplacements. Il est aussi intéressant de noter que ce n'est pas le nombre moyen (bien que cette quantité intervienne dans la définition de Φ et donc dans la définition du cas critique) mais le nombre minimal m

d'enfants qui apparait dans les résultats, phénomène déjà observé par Dembo, Gantert, Peres et Zeitouni [29] dans l'étude des grandes déviations pour la marche simple sur un arbre de Galton-Watson (autrement dit un modèle simple de marche aléatoire en environnement aléatoire sur un arbre, où la particule évolue dans un potentiel donné par une marche aléatoire branchante sur l'axe réel).

Théorème 6.1. *Si $m \geq 2$ et si $M := \text{ess sup} X = \inf\{x \in \mathbb{R}, \mathbb{P}(X > x) > 0\} < +\infty$, alors, pour tout $b > b_c$,*

$$\frac{1}{n^{1/3}} \log \left(-\log \mathbb{P} \left(\min_{|u|=n} \overline{V}(u) > bn^{1/3} \right) \right) \to m^{\frac{(b-b_c)}{M}}.$$

Si $m \geq 2$ et si $x \mapsto -\log \mathbb{P}(X > x)$ est à variation régulière d'index κ, avec $1 < \kappa < +\infty$, alors pour tout $b > b_c$, on a, lorsque $n \to \infty$,

$$\log \mathbb{P} \left(\min_{|u|=n} \overline{V}(u) > bn^{1/3} \right) \sim \log \mathbb{P} \left(X > (b - b_c)n^{1/3} \right) \left(m^{\frac{1}{\kappa-1}} - 1 \right)^{\kappa-1}.$$

Si $\kappa = 1$, la borne supérieure reste vraie. Et on a la borne inférieure suivante :

$$\limsup_{n\to\infty} \frac{\log \mathbb{P} \left(\min_{|u|=n} \overline{V}(u) > bn^{1/3} \right)}{(b - b_c)n^{1/3}} \leq -t_0 m$$

où $t_0 := \sup\{t \in \mathbb{R} : \Lambda_X(t) < +\infty\} \in (0, +\infty]$.

Si $m = 1$, alors il existe des constantes $0 < c_2 \leq c_1 < +\infty$ telles que pour tout $b > b_c$,

$$\begin{aligned} -c_1 &\leq \liminf_{n\to\infty} \frac{\log \mathbb{P} \left(\min_{|u|=n} \overline{V}(u) > bn^{1/3} \right)}{(b - b_c)n^{1/3}}; \\ -c_2 &\geq \limsup_{n\to\infty} \frac{\log \mathbb{P} \left(\min_{|u|=n} \overline{V}(u) > bn^{1/3} \right)}{(b - b_c)n^{1/3}}. \end{aligned}$$

Ce phénomène se révèle facile à interpréter. Lorsque $m \geq 2$, la population croît exponentiellement, et plus il y a d'individus (à une génération N), plus il devient difficile de demander à chaque sous-arbre partant de ces individus de faire un effort pour rattrapper l'écart avec le comportement typique puisque ces sous-arbres sont indépendants les uns des autres. C'est pourquoi l'effort pour rattrapper le retard $(b - b_c)n^{1/3}$ doit porter sur les premières générations (disons les N premières générations), quand il y encore peu d'individus.

Dans le cas exponentiel, on peut prendre $N = 1$ et ainsi faire supporter tout l'effort par la première génération, puis laisser les générations suivantes avoir un comportement typique, car le coût d'un effort important pour un individu est faible, et on obtient immédiatement le résultat (une borne inférieure pour la probabilité étudiée), à cela près que la valeur de la constante c n'est pas toujours optimale (en particulier dans le cas où la queue de ditribution admet des moments de tous ordres, on s'attend à ce que, sous des hypothèses supplémentaires, le comportement se rapproche du cas de type gaussien).

A l'inverse, dans le cas des déplacements bornés supérieurement, même en demandant l'effort maximal possible aux N premières générations, il faut au minimum un nombre de générations N de l'ordre de $n^{1/3}$ pour combler l'écart $(b - b_c)n^{1/3}$ au comportement typique. Le résultat découle de ce que le coût total de cet effort, ainsi que la probabilité que tout se passe bien sur les générations suivantes sont exponentiels en le nombre d'individus à la génération N.

Enfin le cas de type gaussien est intermédiaire entre les deux cités précédemment : il faut un petit nombre de générations pour compenser l'écart, mais le coût total est moindre si on répartit l'effort sur plusieurs générations (avec une contribution diminuant exponentiellement en la génération).

Chapter 2

Survival probability estimates in the critical and subcritical cases

Survival of branching random walks with absorption[1]

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Summary. We consider a branching random walk on $\mathbb{R}$ starting from $x \geq 0$ and with a killing barrier at 0. At each step, particles give birth to b children, which move independently. Particles that enter the negative half-line are killed. In case of almost sure extinction, we find accurate asymptotics for the survival probabilities at time n, when n tends to infinity.

Key words. Branching random walk, survival probability, local limit theorems.

AMS subject classifications. 60J80.

1 Introduction

We consider a branching random walk on $\mathbb{R}$ with an absorbing barrier at the origin. At time n, each individual of the surviving population gives birth to a fixed number of children, which move independently from the position of their father. Particles that enter the negative half-line are immediately killed, and do not have any descendant.

Precisely, take $b \in \mathbb{N}^*$. Let $\mathcal{T}$ be a rooted b-ary tree, with the partial order $v < u$ if v is an ancestor of u (we write $v \leq u$ if $v < u$ or $v = u$), and let $|u|$ denote the generation of u, the generation of the root being zero. We attach i.i.d random variables $(X_u,\, u \in \mathcal{T}, |u| \geq 1)$ (X will denote a generic random variable with the common distribution). For $u \in \mathcal{T}$, we define the position $S(u)$ of u by :

$$S(u) = x + \sum_{v \leq u} X_v \,,$$

where x is the position of the ancestor (the root). The surviving population Z_n at time n is the number of particles that never touched the negative half-line :

$$Z_n := \#\{|u| = n \,:\, S(v) \geq 0 \;\forall\, v \leq u\}.$$

[1]This chapter comes from an article submitted to *Stochastic Processes and their Applications*. It is a joint work with Elie Aidékon.

Any individual u such that $S(u) < 0$ dies, and then has no children. Therefore we are only interested in individuals which have all their ancestors, including themselves, at the right of the barrier 0.

Kesten [61], and recently Harris and Harris [48] worked on the branching Brownian motion with absorption, which is the continuous analog of our problem. We refer to Derrida [31], [32] for a more physical point of view on killed branching random walks. In the setting of branching Brownian motion (without absorption), large deviations probabilities on the speed of the rightmost particle were given by Chauvin and Rouault [25].

The first natural question that arises is whether the population ultimately dies. We introduce

$$\phi(t) := E\left[\mathrm{e}^{tX}\right], \tag{1.1}$$

$$\gamma := \inf_{t\geq 0} E\left[\mathrm{e}^{tX}\right]. \tag{1.2}$$

We make the following assumptions on the step distribution.

- There exists $s > 0$ such that $\phi(s) < \infty$.
- There exists $\nu \in (0, s)$ such that $\phi'(\nu) = 0$.
- The distribution of X is non-lattice.

Under these assumptions, we can show that

(i) if $\gamma \leq 1/b$, there is almost sure extinction,

(ii) if $\gamma > 1/b$, the process survives with positive probability.

This criterion appears in Biggins et al. [15], though the critical case $\gamma = 1/b$ is not treated there. However, a first moment argument easily bridges the gap. We can also see it as a consequence of our results.

Throughout this paper, we focus on the extinction case $\gamma \leq 1/b$. We necessarily have $E[X] \leq 0$ which means that particles are attracted to the barrier, and strongly enough to compensate the reproduction. Moreover, $\gamma = \phi(\nu) = \inf_{t\in\mathbb{R}} \phi(t)$. Define

$$u_n(x) := P^x(Z_n > 0)$$

which is the probability for the process to survive until generation n, starting from x. We already know that this probability tends to zero. The aim of this paper is to estimate the rate of decay of u_n. Our first theorem deals with the subcritical case. Let X_1, $X_2, \ldots$ be i.i.d random variables distributed as X, and define $S_n := S_0 + \sum_{k=1}^n X_k$. Under the probability P^z, $S_0 = z$ almost surely. We introduce $I_k := \inf\{S_j,\, j \leq k\}$ and for any $x \geq 0$ the renewal function associated to $(S_n)_{n\geq 0}$

$$\widetilde{V}(x) := 1 + \sum_{k=1}^{\infty} \gamma^{-k} E^0[\mathrm{e}^{\nu S_k} \mathbb{1}_{\{S_k = I_k \geq -x\}}] . \tag{1.3}$$

Theorem 1.1. *If $\gamma < 1/b$, then there exists a constant $C_1 > 0$ such that for any $x \geq 0$*

$$u_n(x) \sim C_1 \mathrm{e}^{\nu x} \widetilde{V}(x)(b\gamma)^n n^{-3/2}.$$

The proof makes use of the following result for one-dimensional random walks, which is Lemma 3 (ii) of [8].

Theorem A (Bertoin, Doney [8])**.** *There exists a constant $C_2 > 0$ such that for any $x \geq 0$*

$$P^x(I_n \geq 0) \sim C_2 \mathrm{e}^{\nu x} \widetilde{V}(x) n^{-3/2} \gamma^n \ . \tag{1.4}$$

Consequently, the mean population at time n is given by $E^x[Z_n] = C_2 \mathrm{e}^{\nu x} \widetilde{V}(x) n^{-3/2} b^n \gamma^n$. In light of Theorem 1.1, we can therefore state that

$$E^x[Z_n \,|\, Z_n > 0] \to \frac{C_2}{C_1} .$$

Conditionally on non-extinction, the mean population converges to a constant independent of the starting point. Our next result concerns the critical case $\gamma = 1/b$. We find here that the probability to survive is of order smaller than $E^x[Z_n]$, which is in contrast with the subcritical case.

Theorem 1.2. *Suppose that $\gamma = \frac{1}{b}$. Let $\sigma^2 := \phi''(\nu)/\phi(\nu)$. We have, for any $x > 0$,*

$$\log(P^x(Z_n > 0)) \underset{n\to\infty}{\sim} -\left(\frac{3\sigma^2\nu^2\pi^2}{2}\right)^{1/3} n^{1/3} .$$

The paper is organized as follows. Section 2 gives results on local probabilities of one-dimensional random walks conditioned to stay positive, that are used in Section 3 for the proof of Theorem 1.1. In Section 4, we present a result due to Mogul'skii [91] on the probability for a random walk to stay between two curves. We prove Theorem 1.2 in Section 5.

2 The centered random walk conditioned to stay positive

In this section, $(R_n)_{n\geq 0}$ is a centered random walk under some probability $\mathbb{Q}$ such that $E_{\mathbb{Q}}[\mathrm{e}^{\theta(R_1 - R_0)}] < \infty$ for θ in an open set containing 0. For $x \geq 0$, we denote by $\mathbb{Q}^x$ a probability distribution under which we have besides $R_0 = x$ almost surely.

For $z \geq 0$, we define the renewal function $V_R(z)$ by $V_R(0) = 1$ and

$$V_R(z) := \sum_{k \geq 0} \mathbb{Q}^0(R_k = I_k(R) \geq -z)$$

where $I_k(R) := \inf_{\ell \in [0,k]} R_\ell$. We denote by $\tau_0(R)$ the first passage time below zero of the random walk R,

$$\tau_0(R) := \inf\{k \geq 1 \,:\, R_k < 0\}.$$

We introduce the *backward* process $\overleftarrow{R} = (\overleftarrow{R}_k,\, 0 \leq k \leq n)$ which is the random walk with step distribution $-(R_1 - R_0)$. Under $\mathbb{Q}^x$, we suppose that $\overleftarrow{R}_0 = x$ almost surely. We define $I_k(\overleftarrow{R})$, $V_{\overleftarrow{R}}$ and $\tau_0(\overleftarrow{R})$ by analogy with $I_k(R)$, V_R and $\tau_0(R)$. Looking backwards in time, we observe that we have the following equality in distribution

$$(R_{n-k-1} - R_{n-k})_{0 \leq k \leq n} \overset{\text{(dist)}}{=} \left(\overleftarrow{R}_k - \overleftarrow{R}_{k-1}\right)_{0 \leq k \leq n}.$$

Finally, for ease of notation, we will write τ_0 and I_k instead of $\tau_0(R)$ and $I_k(R)$ whenever there is no possible confusion. Our aim is to estimate the probability for the random walk to lie in a small interval at time n without touching 0 before. We mainly use results from Caravenna [24] and Vatutin and Wachtel [107]. However, contrary to the previous works where the random walks start at zero, we want to allow any starting point $x \geq 0$.

2.1 Normal deviations regime

Let $\psi(x) := x\mathrm{e}^{-x^2/2} 1_{\{x \geq 0\}}$, and $\sigma^2 := E_{\mathbb{Q}}[(R_1 - R_0)^2]$. Theorem 1 of [24] says that

$$\sigma\sqrt{n}\mathbb{Q}^0\left(R_n \in [a, a+\delta] \,\middle|\, \tau_0 > n\right) = \delta\psi\left(\frac{a}{\sigma\sqrt{n}}\right) + o_n(1) \tag{2.1}$$

uniformly in $a \in \mathbb{R}_+$ and δ in compact sets of $\mathbb{R}_+$. It is also well-known (see Kozlov [67]) that there exists a constant C_+ such that

$$\mathbb{Q}^0(\tau_0(R) > n) \underset{n \to \infty}{\sim} \frac{C_+}{n^{1/2}}. \tag{2.2}$$

Therefore, we can rewrite (2.1) as

$$\frac{\sigma n}{C_+}\mathbb{Q}^0\left(R_n \in [a, a+\delta],\, \tau_0 > n\right) = \delta\psi\left(\frac{a}{\sigma\sqrt{n}}\right) + o_n(1) \tag{2.3}$$

uniformly in $a \in \mathbb{R}_+$ and δ in compact sets of $\mathbb{R}_+$.

Lemma 2.1. *Let $(d_n)_{n\geq 0}$ be a sequence in $\mathbb{R}_+$ such that $d_n = o\left(\frac{\sqrt{n}}{\log(n)}\right)$. We have*

$$\frac{\sigma n}{C_+ V_R(x)} \mathbb{Q}^x(R_n \in [a, a+\delta],\, \tau_0 > n) = \delta\psi\left(\frac{a}{\sqrt{n}}\right) + o_n(1)$$

uniformly in $(a, x) \in \mathbb{R}_+ \times [0, d_n]$ and δ in compact sets of $\mathbb{R}_+$.

Proof. Let $\Delta > 0$ and $(m_n)_{n\geq 0}$ be a deterministic sequence of integers such that $n >> m_n >> d_n^2$. We define T_n as the first time when the random walk R is equal to the infimum on the interval $[0, n]$

$$T_n := \inf\{k \in [0, n] \,:\, R_k = I_n(R)\}.$$

For any $a > 0$, $\delta > 0$ and $x \in [0, d_n]$, we write

$$\begin{aligned} &\Big|\frac{\sigma n}{C_+ V_R(x)} \mathbb{Q}^x\left(R_n \in [a, a+\delta],\, \tau_0 > n\right) - \delta\psi\left(\frac{a}{\sigma\sqrt{n}}\right)\Big| \\ \leq\; & \frac{\sigma n}{C_+ V_R(x)} \mathbb{Q}^x\left(T_n > m_n,\, R_n \in [a, a+\delta],\, \tau_0 > n\right) \\ &+ \Big|\frac{\sigma n}{C_+ V_R(x)} \mathbb{Q}^x(T_n \leq m_n,\, R_n \in [a, a+\delta],\, \tau_0 > n) - \delta\psi\left(\frac{a}{\sigma\sqrt{n}}\right)\Big|. \end{aligned}$$

Therefore, we have to show that there exist two positive sequences $(\varepsilon_n^{(1)})_{n\geq 0}$ and $(\varepsilon_n^{(2)})_{n\geq 0}$ both with limit 0 such that for any $a > 0$, $\delta \in [0, \Delta]$ and $x \in [0, d_n]$, we have

$$\mathbb{Q}^x(T_n > m_n,\, R_n \in [a, a+\delta],\, \tau_0 > n) \;\leq\; \varepsilon_n^{(1)} \frac{V_R(x)}{n}, \tag{2.4}$$

$$\Big|\mathbb{Q}^x(T_n \leq m_n,\, R_n \in [a, a+\delta],\, \tau_0 > n) - \delta\frac{C_+ V_R(x)}{\sigma n}\psi\left(\frac{a}{\sigma\sqrt{n}}\right)\Big| \;\leq\; \varepsilon_n^{(2)} \frac{V_R(x)}{n}. \tag{2.5}$$

Proof of equation (2.4).
By the Markov property, we have

$$\mathbb{Q}^x(T_n > m_n,\, R_n \in [a, a+\delta],\, \tau_0 > n) = \sum_{k=m_n+1}^{n} E_{\mathbb{Q}}^x\left[L(a - R_k, n-k),\, R_k = I_k(R) \geq 0\right]$$

where for $z \in \mathbb{R}_+$ and ℓ integer, we define

$$L(z, \ell) := \mathbb{Q}^0(R_\ell \in [z, z+\delta],\, \tau_0 > \ell). \tag{2.6}$$

By equation (2.3), there exists a sequence $(\eta_\ell)_{\ell\geq 0}$ tending to 0 such that for any $z \in \mathbb{R}_+$, any $\delta \in [0, \Delta]$ and any integer ℓ,

$$\Big|L(z, \ell) - \frac{C_+\delta}{\sigma(\ell+1)}\psi\left(\frac{z}{\sigma\sqrt{\ell}}\right)\Big| \leq \frac{\eta_\ell}{\ell+1}. \tag{2.7}$$

In particular, $L(z,\ell) \le c_3/(\ell+1)$ for some constant $c_3 > 0$. We deduce that

$$\mathbb{Q}^x(T_n > m_n,\, R_n \in [a, a+\delta],\, \tau_0 > n) \le \sum_{k=m_n+1}^{n} \frac{c_3}{n-k+1} \mathbb{Q}^x(R_k = I_k \ge 0).$$

Looking backwards in time, we see that $\mathbb{Q}^x(R_k = I_k \ge 0) = \mathbb{Q}^0\left(\overleftarrow{R}_k \le x,\, \tau_0(\overleftarrow{R}) > k\right)$. By Lemma 20 of [107], there exists a constant c_4 such that for any $k \ge 1$ and $x \le \sqrt{k}$, we have

$$\mathbb{Q}^0\left(\overleftarrow{R}_k < x,\, \tau_0(\overleftarrow{R}) > k\right) \le c_4(1+x)V_{\overleftarrow{R}}(x)k^{-3/2}.$$

We know (see [106]) that $V_{\overleftarrow{R}}(u)/u$ has a positive limit when u goes to ∞. Therefore we have

$$\mathbb{Q}^0(\overleftarrow{R}_k \le x,\, \tau_0(\overleftarrow{R}) > k) \le c_5(1+x)^2k^{-3/2}$$

for some constant c_5. We deduce that

$$\mathbb{Q}^x(R_k = I_k(R) \ge 0) \le c_5(1+x)^2k^{-3/2}. \tag{2.8}$$

Hence, uniformly in $\delta \in [0, \Delta]$

$$\mathbb{Q}^x(T_n > m_n,\, R_n \in [a, a+\delta],\, \tau_0 > n) \le c_5(1+x)^2 \sum_{k=m_n+1}^{n} \frac{c_3}{n-k+1} k^{-3/2}.$$

We see that $\sum_{k=m_n+1}^{n/2} \frac{1}{n-k+1}k^{-3/2} \le c_6 m_n^{-1/2}/n$ and $\sum_{k=n/2}^{n} \frac{1}{n-k+1}k^{-3/2} \le c_7 n^{-3/2}\log(n)$. It yields that

$$\begin{aligned}\mathbb{Q}^x(T_n > m_n,\, R_n \in [a, a+\delta],\, \tau_0 > n) &\le c_3c_5(1+x)^2(c_6(m_n^{-1/2}/n) + c_7n^{-3/2}\log(n)) \\ &\le c_8\frac{V_R(x)}{n} d_n(m_n^{-1/2} + n^{-1/2}\log(n))\end{aligned}$$

since $V_R(x) \ge c_9(1+x)$ for some $c_9 > 0$ ($V_R(x) \ge 1$ for any $x \ge 0$ and $V_R(x)/x$ converges at infinity to a positive limit from [106]). Then we choose $\varepsilon_n^{(1)} := c_8 d_n(m_n^{-1/2} + n^{-1/2}\log(n))$ to complete the proof of (2.4).

Proof of equation (2.5)

We write as before

$$\mathbb{Q}^x(T_n \le m_n,\, R_n \in [a, a+\delta],\, \tau_0 > n) = \sum_{k=0}^{m_n} E_{\mathbb{Q}}^x\left[L(a - R_k, n-k),\, R_k = I_k \ge 0\right]$$

where $L(z,\ell)$ is defined in (2.6). By the definition of $V_R(x)$, we have

$$\Big|\mathbb{Q}^x(T_n \le m_n,\, R_n \in [a, a+\delta],\, \tau_0 > n) - \delta \frac{V_R(x)C_+}{\sigma n}\psi\left(\frac{a}{\sigma\sqrt{n}}\right)\Big| \tag{2.9}$$

$$\le \sum_{k=0}^{m_n} E_{\mathbb{Q}}^x\left[\Big|L(a-R_k, n-k) - \frac{\delta C_+}{\sigma n}\psi\left(\frac{a}{\sigma\sqrt{n}}\right)\Big|,\, R_k = I_k \ge 0\right] \tag{2.10}$$

$$+ \frac{\delta C_+ \psi\left(\frac{a}{\sigma\sqrt{n}}\right)}{\sigma n}\sum_{k>m_n}\mathbb{Q}^x(R_k = I_k \ge 0).$$

By equation (2.8), we have

$$\sum_{k>m_n}\mathbb{Q}^x(R_k = I_k > 0) \le c_5(1+x)^2\sum_{k>m_n} k^{-3/2} \le c_{10} d_n V_R(x) m_n^{-1/2}.$$

It follows that for any $a \in \mathbb{R}_+$, $x \in [0, d_n]$ and $\delta \in [0, \Delta]$,

$$\frac{\delta C_+ \psi\left(\frac{a}{\sigma\sqrt{n}}\right)}{\sigma n}\sum_{k>m_n}\mathbb{Q}^x(R_k = I_k \ge 0) \le c_{10}\frac{V_R(x)}{n}\left(\frac{\Delta C_+\|\psi\|_\infty d_n}{m_n^{1/2}\sigma}\right) =: \frac{V_R(x)}{n}\eta_n^{(1)} \tag{2.11}$$

where $\eta_n^{(1)} = o_n(1)$ by our choice of m_n. On the other hand

$$\sum_{k=0}^{m_n} E_{\mathbb{Q}}^x\left[\Big|L(a-R_k, n-k) - \frac{\delta C_+}{\sigma n}\psi\left(\frac{a}{\sigma\sqrt{n}}\right)\Big|, R_k = I_k \ge 0\right] \tag{2.12}$$

$$\le \sum_{k=0}^{m_n} E_{\mathbb{Q}}^x\left[\Big|L(a-R_k, n-k) - \frac{\delta C_+}{\sigma(n-k)}\psi\left(\frac{a-R_k}{\sigma\sqrt{n-k}}\right)\Big|,\, R_k = I_k \ge 0\right]$$

$$+(\delta C_+/\sigma)\sum_{k=0}^{m_n} E_{\mathbb{Q}}^x\left[\Big|\frac{1}{n-k}\psi\left(\frac{a-R_k}{\sigma\sqrt{n-k}}\right) - \frac{1}{n}\psi\left(\frac{a}{\sigma\sqrt{n}}\right)\Big|,\, R_k = I_k \ge 0\right].$$

By equation (2.7), we have

$$\sum_{k=0}^{m_n} E_{\mathbb{Q}}^x\left[\Big|L(a-R_k, n-k) - \frac{\delta C_+}{\sigma(n-k)}\psi\left(\frac{a-R_k}{\sigma\sqrt{n-k}}\right)\Big|,\, R_k = I_k \ge 0\right]$$

$$\le \sum_{k=0}^{m_n}\mathbb{Q}^x(R_k = I_k \ge 0)\frac{1}{n-k}\eta_{n-k}$$

$$\le \frac{V_R(x)}{n}\eta_n^{(2)}$$

with $\eta_n^{(2)} := \frac{n}{n-m_n}\sup_{\ell \ge n-m_n}\eta_\ell$. The analysis of the function ψ shows that

$$\sup_{y\in[a-x,a]}\Big|\frac{1}{n-k}\psi\left(\frac{y}{\sigma\sqrt{n-k}}\right) - \frac{1}{n}\psi\left(\frac{a}{\sigma\sqrt{n}}\right)\Big| = o_n(1/n)$$

uniformly in $(a,x) \in \mathbb{R}_+ \times [0,d_n]$. We deduce the existence of $(\eta_n^{(3)})_{n\geq 0}$ going to 0 such that

$$\begin{aligned}
&\sum_{k=0}^{m_n} E_{\mathbb{Q}}^x\left[\left|\frac{1}{n-k}\psi\left(\frac{a-R_k}{\sigma\sqrt{n-k}}\right)-\frac{1}{n}\psi\left(\frac{a}{\sigma\sqrt{n}}\right)\right|,\, R_k=I_k\geq 0\right]\\
\leq\ &\sum_{k=0}^{m_n}\mathbb{Q}^x(R_k=I_k\geq 0)\sup_{y\in[a,a+x]}\left|\frac{1}{n-k}\psi\left(\frac{y}{\sigma\sqrt{n-k}}\right)-\frac{1}{n}\psi\left(\frac{a}{\sigma\sqrt{n}}\right)\right|\\
\leq\ &\frac{1}{n}\eta_n^{(3)}\sum_{k=1}^{m_n}\mathbb{Q}^x(S_k=I_k\geq 0)\leq\frac{1}{n}\eta_n^{(3)}V_R(x).
\end{aligned}$$

Thus, equation (2.12) becomes

$$\sum_{k=0}^{m_n} E_{\mathbb{Q}}^x\left[\left|L(a-R_k,n-k)-\frac{\delta C}{\sigma n}\psi\left(\frac{a}{\sigma\sqrt{n}}\right)\right|,\, R_k=I_k\geq 0\right]\leq\frac{V_R(x)}{n}(\eta_n^{(2)}+(\delta C_+/\sigma)\eta_n^{(3)}).$$

This combined with equations (2.9) and (2.11) implies that

$$\begin{aligned}
&\left|\mathbb{Q}^x(T_n\leq m_n,\, R_n\in[a,a+\delta],\,\tau_0>n)-\delta\frac{V_R(x)C_+}{\sigma n}\psi\left(\frac{a}{\sigma\sqrt{n}}\right)\right|\\
\leq\ &\frac{V_R(x)}{n}(\eta_n^{(1)}+\eta_n^{(2)}+(\delta C_+/\sigma)\eta_n^{(3)}).
\end{aligned}$$

Hence equation (2.5) holds with $\varepsilon_n^{(2)} := \eta_n^{(1)}+\eta_n^{(2)}+(\delta C_+/\sigma)\eta_n^{(3)}$. □

A result from Iglehart [54] says that under $\mathbb{Q}^0$, the random walk $R_n/(\sigma\sqrt{n})$ conditioned to stay positive converges to the Rayleigh distribution. We use this result to prove a more general result :

Lemma 2.2. *Let $(d_n)_{n\geq 0}$ be a sequence such that $d_n = o(\sqrt{n})$. For any bounded continuous function f, we have*

$$E_{\mathbb{Q}}^x\left[f\left(\frac{R_n}{\sigma\sqrt{n}}\right),\,\tau_0>n\right]=\frac{C_+V_R(x)}{\sqrt{n}}\left(\int_0^\infty f(u)\psi(u)du+o_n(1)\right)\tag{2.13}$$

uniformly in $x\in[0,d_n]$.

Proof. Suppose first that f is a Lipschitz function with compact support. Write

$$E_{\mathbb{Q}}^x\left[f\left(\frac{R_n}{\sigma\sqrt{n}}\right),\,\tau_0>n\right]=\sum_{k=0}^{n}E_{\mathbb{Q}}^x\left[a_n(R_k,k),\,R_k=I_k\geq 0\right]$$

with

$$a_n(z,k):=E_{\mathbb{Q}}^0\left[f\left(\frac{R_{n-k}+z}{\sigma\sqrt{n}}\right),\,\tau_0>n-k\right].\tag{2.14}$$

We have

$$a_n(z,k) \le \|f\|_\infty \mathbb{Q}^0(\tau_0 > n-k) \le \frac{c_{11}}{\sqrt{n-k+1}}.$$

Let $(m_n)_{n\ge 0}$ be a sequence of integers such that $d_n^2 << m_n << n$. We deduce that

$$\begin{aligned}\sum_{k=m_n+1}^{n} E_{\mathbb{Q}}^x\left[a_n(R_k,k),\, R_k = I_k \ge 0\right] &\le c_{11}\sum_{k=m_n+1}^{n} \frac{1}{\sqrt{n-k+1}}\mathbb{Q}^x(R_k = I_k \ge 0)\\ &\le c_{11}c_5(1+x)^2 \sum_{k=m_n+1}^{n} \frac{1}{\sqrt{n-k+1}} k^{-3/2}\end{aligned}$$

by equation (2.8). We see that $\sum_{k=m_n+1}^{n} \frac{1}{\sqrt{n-k+1}} k^{-3/2} = O\left(\frac{1}{\sqrt{n m_n}}\right)$, hence

$$\sum_{k=m_n+1}^{n} E_{\mathbb{Q}}^x\left[a_n(R_k,k),\, R_k = I_k \ge 0\right] \le \frac{V_R(x)}{\sqrt{n}}\left(c_{12}\frac{d_n}{\sqrt{m_n}}\right). \tag{2.15}$$

On the other hand, since f is Lipschitz, we check that there exists c_{13} such that

$$\sup_{y\in\mathbb{R}, z\in[0,x]} \left| f\left(\frac{y+z}{\sqrt{n}}\right) - f\left(\frac{y}{\sqrt{n-k}}\right)\right| \le c_{13}\left(\frac{x}{\sqrt{n}} + \frac{k}{n^{3/2}}\right).$$

From the expression (2.14) of $a_n(z,k)$, we have for any $z \in [0,x]$,

$$\left|a_n(z,k) - E_{\mathbb{Q}}^0\left[f\left(\frac{R_{n-k}}{\sigma\sqrt{n-k}}\right),\, \tau_0 > n-k\right]\right| \le c_{13}\left(\frac{x}{\sqrt{n}} + \frac{k}{n^{3/2}}\right)\mathbb{Q}^0(\tau_0 > n-k). \tag{2.16}$$

We know that $\frac{R_n}{\sigma\sqrt{n}}$ conditionally on $\tau_0 > n$ converges under $\mathbb{Q}^0$ to the Rayleigh distribution (see [54]). It implies that there exists $(\eta_k)_{k\ge 0}$ tending to zero such that

$$\left|E_{\mathbb{Q}}^0\left[f\left(\frac{R_{n-k}}{\sigma\sqrt{n-k}}\right),\, \tau_0 > n-k\right] - \mathbb{Q}^0(\tau_0 > n-k)\int_0^\infty f(u)\psi(u)du\right| \le \eta_{n-k}\mathbb{Q}^0(\tau_0 > n-k). \tag{2.17}$$

Let $\varepsilon > 0$. For n large enough and $k \le m_n$, we have from (2.2) $|\mathbb{Q}^0(\tau_0 > n-k) - \frac{C_+}{\sqrt{n}}| \le \frac{\varepsilon}{\sqrt{n}}$.

Combined with (2.16) and (2.17), this gives

$$\begin{aligned}
&\left|a_n(z,k) - \frac{C_+}{\sqrt{n}}\int_0^\infty f(u)\psi(u)du\right| \\
\leq\; & c_{13}\left(\frac{x}{\sqrt{n}} + \frac{k}{n^{3/2}}\right)\mathbb{Q}^0(\tau_0 > n-k) + \eta_{n-k}\mathbb{Q}^0(\tau_0 > n-k) + \|f\|_\infty \frac{\varepsilon}{\sqrt{n}} \\
\leq\; & c_{14}\left(\frac{x}{n} + \frac{k}{n^2} + \frac{\varepsilon}{\sqrt{n}}\right) \\
\leq\; & c_{14}\left(\frac{d_n}{n} + \frac{m_n}{n^2} + \frac{\varepsilon}{\sqrt{n}}\right) \\
\leq\; & 2c_{14}\frac{\varepsilon}{\sqrt{n}}
\end{aligned}$$

for n greater than some n_1, $k \leq m_n$, $x \in [0, d_n]$ and $z \in [0, x]$. We use this inequality for every $k = 0 \cdots, m_n$ and we obtain

$$\begin{aligned}
\sum_{k=0}^{m_n}\left|E_{\mathbb{Q}}^x\left[a_n(R_k,k),\, R_k = I_k > 0\right] - \frac{C_+}{\sqrt{n}}\int_0^\infty f(u)\psi(u)du\right| \;&\leq\; 2c_{14}\frac{\varepsilon}{\sqrt{n}}\sum_{k=0}^{m_n}\mathbb{Q}^x(R_k = I_k > 0) \\
&\leq\; 2c_{14}\frac{\varepsilon}{\sqrt{n}}V_R(x).
\end{aligned}$$

Together with (2.15), it yields that

$$\left|E_{\mathbb{Q}}^x\left[f\left(\frac{R_n}{\sigma\sqrt{n}}\right),\, \tau_0 > n\right] - \frac{C_+}{\sqrt{n}}\int_0^\infty f(u)\psi(u)du\right| \leq \frac{V_R(x)}{\sqrt{n}}(c_{12}\frac{d_n}{\sqrt{m_n}} + 2c_{14}\varepsilon)$$

which implies equation (2.13). To complete the proof, we need to check that for any $\varepsilon > 0$, there exists A large enough such that

$$\sup_{x\in[0,d_n]}\mathbb{Q}^x\left(\frac{R_n}{\sigma\sqrt{n}} > A,\, \tau_0 > n\right) \leq V_R(x)\frac{\varepsilon}{\sqrt{n}}.$$

Equivalently, we need to check that for any $x \in [0, d_n]$

$$\sum_{k=0}^{n} E_{\mathbb{Q}}^x\left[\mathbb{Q}^0\left(\frac{R_{n-k}+z}{\sigma\sqrt{n}} > A,\, \tau_0 > n-k\right)_{z=R_k},\, R_k = I_k \geq 0\right] \leq V_R(x)\frac{\varepsilon}{\sqrt{n}}.$$

For $k \geq m_n$, we already know from (2.15) that

$$\sum_{k=m_n+1}^{n}\mathbb{Q}^x(R_k = I_k > 0) \leq \frac{V_R(x)}{\sqrt{n}}\left(c_{12}\frac{d_n}{\sqrt{m_n}}\right) \leq \frac{V_R(x)}{\sqrt{n}}\varepsilon$$

for n greater than some n_1 and any $x \in [0, d_n]$. Let A such that $\mathbb{Q}^0(\frac{R_n}{\sigma\sqrt{n}} > A-1) \le \frac{\varepsilon}{\sqrt{n}}$ for n large enough. We observe that for any $z \in [0, d_n]$,

$$\begin{aligned}\mathbb{Q}^0\left(\frac{R_{n-k}+z}{\sigma\sqrt{n}} > A,\, \tau_0 > n-k\right) &\le \mathbb{Q}^0\left(\frac{R_{n-k}}{\sigma\sqrt{n-k}} > A - \frac{d_n}{\sigma\sqrt{n}},\, \tau_0 > n-k\right)\\ &\le \frac{\varepsilon}{\sqrt{n-k}} \le 2\frac{\varepsilon}{\sqrt{n}}\end{aligned}$$

for n large enough and $k \le m_n$. It yields that

$$\sum_{k=0}^{m_n} E_{\mathbb{Q}}^x\left[\mathbb{Q}^0\left(\frac{R_{n-k}+z}{\sigma\sqrt{n}} > A,\, \tau_0 > n-k\right)_{z=R_k},\, R_k = I_k \ge 0\right] \le 2\varepsilon\frac{V_R(x)}{\sqrt{n}}.$$

Therefore, there exists n_2 such that for any $n \ge n_2$, and any $x \in [0, d_n]$,

$$\sum_{k=0}^{n} E_{\mathbb{Q}}^x\left[\mathbb{Q}^0\left(\frac{R_{n-k}+z}{\sigma\sqrt{n}} > A,\, \tau_0 > n-k\right)_{z=R_k},\, R_k = I_k \ge 0\right] \le (3\varepsilon)\frac{V_R(x)}{\sqrt{n}}$$

which completes the proof. $\square$

2.2 Small deviations regime

We are interested in regimes where the random walk is close to the origin. This has been investigated in [107] in the case of a starting point $x = 0$.

We recall that C_+ was defined in (2.2). Similarly, let C_- be the positive constant such that

$$\mathbb{Q}^x(\tau_0(\overleftarrow{R}) > n) \underset{n\to\infty}{\sim} \frac{C_-}{\sqrt{n}}.$$

Equivalently,

$$\mathbb{Q}^x(\max_{1\le k\le n} R_k \le 0) \underset{n\to\infty}{\sim} \frac{C_-}{\sqrt{n}}.$$

Lemma 2.3. *Let d_n be a sequence in $\mathbb{R}_+$ such that $d_n = o\left(\frac{\sqrt{n}}{\log(n)}\right)$ and $\delta > 0$. We have*

$$\mathbb{Q}^x(R_n \in [z, z+\delta), \tau_0 > n) \underset{n\to\infty}{\sim} \frac{C_- C_+}{2\sigma\sqrt{2\pi}} \frac{V_R(x)}{n^{3/2}} \int_z^{z+\delta} V_{\overleftarrow{R}}(u)du$$

uniformly in $(x, z) \in [0, d_n] \times [0, d_n]$.

Proof. For ease of notation, we prove the theorem for n even. Let $\delta > 0$, and $\delta' \in [0, \delta]$. By the Markov property, we have for any $z \in \mathbb{R}_+$,

$$\mathbb{Q}^x(R_n \in [z, z+\delta'),\, \tau_0 > n) = E_{\mathbb{Q}}^x\left[\mathbb{Q}^{R_{n/2}}\left(R_{n/2} \in [z, z+\delta'],\, \tau_0 > n/2\right),\, \tau_0 > n/2\right]. \tag{2.18}$$

Looking backwards in time, we see that for any $y > 0$ and any integer k,

$$\mathbb{Q}^y(R_k \in [z, z+\delta'),\, \tau_0 > k) \leq \mathbb{Q}^{z+\delta'}\left(\overleftarrow{R}_k \in [y, y+\delta'),\, \tau_0(\overleftarrow{R}) > k\right).$$

By Lemma 2.1 applied to the backward process $\overleftarrow{R}$, we deduce that uniformly in $(y, z) \in \mathbb{R}_+ \times [0, d_n]$ and in $\delta' \in [0, \delta]$, we have

$$\mathbb{Q}^y(R_k \in [z, z+\delta'),\, \tau_0 > k) \leq \frac{C_- V_{\overleftarrow{R}}(z+\delta')}{\sigma k}\left(\delta' \psi\left(\frac{y}{\sigma\sqrt{k}}\right) + o_k(1)\right).$$

By equation (2.18), it yields that

$$\mathbb{Q}^x(R_n \in [z, z+\delta'),\, \tau_0 > n) \leq \frac{C_- V_{\overleftarrow{R}}(z+\delta')}{\sigma(n/2)} E_{\mathbb{Q}}^x\left[\delta' \psi\left(\frac{R_{n/2}}{\sigma\sqrt{n/2}}\right) + o_n(1),\, \tau_0 > n/2\right].$$

Lemma 2.2 implies that, uniformly in $x \leq d_n$,

$$\begin{aligned} E_{\mathbb{Q}}^x\left[\psi\left(\frac{R_{n/2}}{\sigma\sqrt{n/2}}\right),\, \tau_0 > (n/2)\right] &= C_+ \int_0^\infty \psi(u)^2 du \frac{V_R(x)}{\sqrt{n/2}}(1 + o_n(1)) \\ &= \frac{C_+\sqrt{\pi}}{4}\frac{V_R(x)}{\sqrt{n/2}}(1 + o_n(1)). \end{aligned}$$

It follows that uniformly in $(x, z) \in [0, d_n] \times [0, d_n]$ and in $\delta' \in [0, \delta]$,

$$\begin{aligned} \mathbb{Q}^x(R_n \in [z, z+\delta'),\, \tau_0 > n) &\leq \frac{C_- C_+\sqrt{\pi}}{4}\frac{V_R(x)}{(n/2)^{3/2}} V_{\overleftarrow{R}}(z+\delta')(\delta' + o_n(1)) \\ &= \frac{C_- C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}} V_{\overleftarrow{R}}(z+\delta')(\delta' + o_n(1)). \end{aligned}$$

We show similarly that

$$\mathbb{Q}^x(R_n \in [z, z+\delta'),\, \tau_0 > n) \geq \frac{C_- C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}} V_{\overleftarrow{R}}(z)(\delta' + o_n(1)).$$

More explicitly, this means that we can find a sequence $(\varepsilon_n)_{n\geq 0}$ tending to zero such that for any $(x, z) \in [0, d_n] \times [0, d_n]$, and any $\delta' \in [0, \delta]$, we have

$$\mathbb{Q}^x(R_n \in [z, z+\delta'),\, \tau_0 > n) \leq \frac{C_- C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}} V_{\overleftarrow{R}}(z+\delta')\delta'(1 + \varepsilon_n/\delta'), \tag{2.19}$$

$$\mathbb{Q}^x(R_n \in [z, z+\delta'),\, \tau_0 > n) \geq \frac{C_- C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}} V_{\overleftarrow{R}}(z)\delta'(1 - \varepsilon_n/\delta'). \tag{2.20}$$

We can rewrite (2.19) as

$$\mathbb{Q}^x(R_n \in [z, z+\delta'),\, \tau_0 > n) \leq \Big\{\frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}}\Big\} V_{\overleftarrow{R}}(z)\delta'(1+\varepsilon_n/\delta')(1+m_{\delta'}(z))$$

where

$$m_{\delta'}(z) := \frac{V_{\overleftarrow{R}}(z+\delta') - V_{\overleftarrow{R}}(z)}{V_{\overleftarrow{R}}(z)}. \tag{2.21}$$

With (2.20), it follows that we have for any $(x,z) \in [0,d_n] \times [0,d_n]$ and $\delta' \in [0,\delta]$

$$\begin{aligned} &\Big|\mathbb{Q}^x(R_n \in [z, z+\delta'),\, \tau_0 > n) - \frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}} V_{\overleftarrow{R}}(z)\delta'\Big| \\ \leq\ & \Big\{\frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}}\Big\} V_{\overleftarrow{R}}(z)\delta'(\varepsilon_n/\delta' + m_{\delta'}(z) + \varepsilon_n m_{\delta'}(z)/\delta'). \end{aligned} \tag{2.22}$$

Let $\varepsilon > 0$. Let $Z > 0$ be such that $\frac{V_{\overleftarrow{R}}(z+\delta)}{V_{\overleftarrow{R}}(z)} - 1 \leq \varepsilon$ for any $z \geq Z$. Using (2.22) with $\delta' = \delta$ implies that for any $z \in [Z, d_n]$ and any $x \in [0, d_n]$, we have

$$\begin{aligned} &\Big|\mathbb{Q}^x(R_n \in [z, z+\delta),\, \tau_0 > n) - \frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}} V_{\overleftarrow{R}}(z)\delta\Big| \\ \leq\ & \Big\{\frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}}\Big\} V_{\overleftarrow{R}}(z)\delta(\varepsilon_n/\delta + \varepsilon + \varepsilon_n\varepsilon/\delta) \\ \leq\ & \Big\{\frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}}\Big\} V_{\overleftarrow{R}}(z)\delta(2\varepsilon) \end{aligned}$$

for n greater than some constant n_1. We want to replace $V_{\overleftarrow{R}}(z)\delta$ by the integral $\int_z^{z+\delta} V_{\overleftarrow{R}}(u)du$. We observe that for any $z \geq Z$,

$$\begin{aligned} \Big|V_{\overleftarrow{R}}(z)\delta - \int_z^{z+\delta} V_{\overleftarrow{R}}(u)du\Big| &\leq \int_z^{z+\delta} (V_{\overleftarrow{R}}(u) - V_{\overleftarrow{R}}(z))du \\ &\leq \delta(V_{\overleftarrow{R}}(z+\delta) - V_{\overleftarrow{R}}(z)) \\ &\leq \delta V_{\overleftarrow{R}}(z)\varepsilon. \end{aligned}$$

It yields that for $n \geq n_1$, $x \in [0, d_n]$ and $z \in [Z, d_n]$,

$$\begin{aligned} &\Big|\mathbb{Q}^x(R_n \in [z, z+\delta),\, \tau_0 > n) - \frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}} \int_z^{z+\delta} V_{\overleftarrow{R}}(u)du\Big| \\ \leq\ & \Big\{\frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}}\Big\} \delta V_{\overleftarrow{R}}(z)(3\varepsilon) \\ \leq\ & \frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}} \int_z^{z+\delta} V_{\overleftarrow{R}}(u)du(3\varepsilon) \end{aligned} \tag{2.23}$$

the last line coming from the fact that $V_{\overleftarrow{R}}$ is nondecreasing. It remains to deal with the case $z \in [0, Z]$. For $h > 0$, we define the compact set $S(h) := \{z \in [0, Z + \delta] : V_{\overleftarrow{R}}(u+) - V_{\overleftarrow{R}}(u-) \le \varepsilon \ \forall u \in [z - h, z + h]\}$. We notice that $[0, Z + \delta]\backslash S(h)$ can be described as a finite union of intervals $\cup_i (z_i - h, z_i + h)$ where $(z_i)_i$ are the points in $[0, Z + \delta]$ such that $V_{\overleftarrow{R}}(z_i+) - V_{\overleftarrow{R}}(z_i-) > \varepsilon$. Therefore, in view of (2.19), we can take $h > 0$ small enough to have for n large enough (say n greater than n_2),

$$\mathbb{Q}^x(R_n \in [0, Z + \delta]\backslash S(h),\, \tau_0 > n) \le \frac{C_- C_+ \sqrt{\pi}}{\sigma\sqrt{2}} \frac{V_R(x)}{n^{3/2}}, \tag{2.24}$$

$$\int_{[0,Z+\delta]\backslash S(h)} V_{\overleftarrow{R}}(u)du \le \delta\varepsilon. \tag{2.25}$$

We observe that $\limsup_{\eta\to 0} \sup_{z\in S(h)} m_\eta(z) \le \varepsilon$ by compacity of $S(h)$. Let n_3 be such that $\sup_{z\in S(h)} m_{2\sqrt{\varepsilon_n}}(z) \le \varepsilon$ for any $n \ge n_3$. Equation (2.22) implies that for any $\delta' \in [\sqrt{\varepsilon_n}, 2\sqrt{\varepsilon_n}]$, for any $z \in S(h)$, and any $n \ge n_3$,

$$\begin{aligned}
&\left|\mathbb{Q}^x(R_n \in [z, z + \delta'),\, \tau_0 > n) - \frac{C_- C_+ \sqrt{\pi}}{\sigma\sqrt{2}} \frac{V_R(x)}{n^{3/2}} V_{\overleftarrow{R}}(z)\delta'\right| \\
&\le \left\{\frac{C_- C_+ \sqrt{\pi}}{\sigma\sqrt{2}} \frac{V_R(x)}{n^{3/2}}\right\} V_{\overleftarrow{R}}(z)\delta'(\sqrt{\varepsilon_n} + \varepsilon + \varepsilon\sqrt{\varepsilon_n}) \\
&\le \left\{\frac{C_- C_+ \sqrt{\pi}}{\sigma\sqrt{2}} \frac{V_R(x)}{n^{3/2}}\right\} V_{\overleftarrow{R}}(z)\delta'(2\varepsilon) \qquad (2.26)
\end{aligned}$$

for n greater than some n_4. As before, we replace $V_{\overleftarrow{R}}(z)\delta'$ by the integral $\int_z^{z+\delta'} V_{\overleftarrow{R}}(u)du$ and equation (2.26) becomes

$$\begin{aligned}
&\left|\mathbb{Q}^x(R_n \in [z, z + \delta'),\, \tau_0 > n) - \frac{C_- C_+ \sqrt{\pi}}{\sigma\sqrt{2}} \frac{V_R(x)}{n^{3/2}} \int_z^{z+\delta'} V_{\overleftarrow{R}}(u)du\right| \qquad (2.27) \\
&\le \frac{C_- C_+ \sqrt{\pi}}{\sigma\sqrt{2}} \frac{V_R(x)}{n^{3/2}} \int_z^{z+\delta'} V_{\overleftarrow{R}}(u)du(3\varepsilon).
\end{aligned}$$

For any $z_1 < z_2$ in $[0, Z + \delta]$ such that $[z_1, z_2] \subset S(h)$, decomposing the interval $[z_1, z_2)$ in small intervals of length between $\sqrt{\varepsilon_n}$ and $2\sqrt{\varepsilon_n}$ yields that for any $n \ge n_4$

$$\begin{aligned}
&\left|\mathbb{Q}^x(R_n \in [z_1, z_2),\, \tau_0 > n) - \frac{C_- C_+ \sqrt{\pi}}{\sigma\sqrt{2}} \frac{V_R(x)}{n^{3/2}} \int_{z_1}^{z_2} V_{\overleftarrow{R}}(u)du\right| \\
&\le \frac{C_- C_+ \sqrt{\pi}}{\sigma\sqrt{2}} \frac{V_R(x)}{n^{3/2}} \int_{z_1}^{z_2} V_{\overleftarrow{R}}(u)du(3\varepsilon).
\end{aligned}$$

Beware that the same inequality is true for non-degenerate intervals of the form $(,]$, $[,]$, or $(,)$ (we can see it by taking slightly larger or smaller intervals of the form $[,)$ and then using the continuity of the integral $\int_{z_1}^{z_2}$). Therefore, we have for any $z \in [0, Z]$ and $n \geq n_4$

$$\begin{aligned}
&\Big|\mathbb{Q}^x(R_n \in [z, z+\delta) \cap S(h),\, \tau_0 > n) - \frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}}\int_{[z,z+\delta)\cap S(h)} V_{\overleftarrow{R}}(u)du\Big| \\
\leq\ & \frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}}\int_{[z,z+\delta)\cap S(h)} V_{\overleftarrow{R}}(u)du(3\varepsilon).
\end{aligned}$$

By our choice of h (see equations (2.24) and (2.25)), we have

$$\begin{aligned}
\mathbb{Q}^x(R_n \in [z, z+\delta] \cap S(h)^c,\, \tau_0 > n) &\leq \mathbb{Q}^x(R_n \in [0, Z+\delta]\backslash S(h),\, \tau_0 > n) \\
&\leq \frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}}\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\int_{[z,z+\delta]\cap S(h)^c} V_{\overleftarrow{R}}(u)du &\leq \int_{[0,Z+\delta]\backslash S(h)} V_{\overleftarrow{R}}(u)du \\
&\leq \delta\varepsilon.
\end{aligned}$$

It yields that for any $x \in [0, d_n]$, any $z \in [0, Z]$ and any $n \geq n_4$,

$$\begin{aligned}
&\Big|\mathbb{Q}^x(R_n \in [z, z+\delta),\, \tau_0 > n) - \frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}}\int_z^{z+\delta} V_{\overleftarrow{R}}(u)du\Big| \\
\leq\ & \Big\{\frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}}\Big\}\int_z^{z+\delta} V_{\overleftarrow{R}}(u)du(5\varepsilon).
\end{aligned}$$

In view of (2.23), we have for $n \geq \max\{n_1, n_4\}$, and any $(x, z) \in [0, d_n] \times [0, d_n]$

$$\begin{aligned}
&\Big|\mathbb{Q}^x(R_n \in [z, z+\delta),\, \tau_0 > n) - \frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}}\int_z^{z+\delta} V_{\overleftarrow{R}}(u)du\Big| \\
\leq\ & \Big\{\frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\frac{V_R(x)}{n^{3/2}}\Big\}\int_z^{z+\delta} V_{\overleftarrow{R}}(u)du(5\varepsilon)
\end{aligned}$$

which completes the proof. □

3 The subcritical case

Recall that $S_n = S_0 + X_1 + X_2 + \cdots + X_n$ is the one-dimensional random walk associated to the branching process. We denote by

$$\tau_0 := \inf\{k \geq 1 \,:\, S_k < 0\}$$

the first passage time to 0 of the random walk S. Since $E[X_1] < 0$, we know that S_n drifts to $-\infty$ and $\tau_0 < \infty$ almost surely. Notice also that for any vertex $|u| = n$, $S(u)$ is distributed as S_n.

Let for $h \in]0, 1]$:

$$B(h) := \frac{1 - (1-h)^b}{bh}.$$

Our first lemma shows a recurrence formula for $u_n(x) := P^x(Z_n > 0)$.

Lemma 3.1. *For any $x \in \mathbb{R}$ and $n \geq 1$, we have*

$$u_n(x) = b\mathbb{1}_{\{x\geq 0\}} E^x[u_{n-1}(S_1)] \, B\left(E^x[u_{n-1}(S_1)]\right).$$

Proof. Firstly, we obviously have $u_n(x) = 0$ if $x < 0$. Therefore take $x \geq 0$. We write that the process survives (until the n^{th} generation) if and only if at least one of the individuals in the first generation has a descendant in the n^{th} generation.

$$u_n(x) = E^x\left[\left(1 - \prod_{i=1}^{b}(1 - \mathbb{1}_{A_i})\right)\right],$$

with A_i the event {the i^{th} individual in the first generation has a descendant in the n^{th} generation}. Using the branching property of the process, one gets that the events $\{(A_i, i = 1, \ldots, b)\}$ are independent and have the same probability equal to $E^x[u_{n-1}(S_1)]$. Put this in the preceding equation to obtain

$$u_n(x) = E^x\left[1 - (1 - E[u_{n-1}(S_1)])^b\right].$$

The conclusion follows from the definition of B. $\square$

Define for any $n \geq 1$ and $x \in \mathbb{R}$:

$$w_n(x) := B\left(E^x[u_{n-1}(S_1)]\right).$$

This allows us to rewrite the lemma as follows :

$$u_n(x) = b\mathbb{1}_{\{x\geq 0\}} E^x[u_{n-1}(S_1)] \, w_n(x). \tag{3.1}$$

For future use, notice that

$$\begin{aligned} 1 - w_n(x) &\leq \frac{b-1}{2} E^x[u_{n-1}(S_1)] \\ &\leq \frac{b-1}{2} u_n(x) =: c_3 u_n(x). \end{aligned} \tag{3.2}$$

Lemma 3.2. *For any $n \geq 0$ and $x \in \mathbb{R}$, we have*

$$u_n(x) = \mathbb{1}_{\{x \geq 0\}} b^n E^x \left[\mathbb{1}_{\{\tau_0 > n\}} \prod_{k=1}^{n} w_k(S_{n-k}) \right] \tag{3.3}$$

Proof. We proceed by induction. The case $n = 0$ is easy since $u_0(x) = \mathbb{1}_{\{x \geq 0\}}$. Now suppose $n \geq 1$ and $x \geq 0$. By equation (3.1), we have

$$u_n(x) = bE^x \left[u_{n-1}(S_1) \right] w_n(x).$$

Applying the recurrence hypothesis to $u_{n-1}(S_1)$ gives :

$$\begin{aligned} u_n(x) &= b^n E^x \left[\mathbb{1}_{\{S_1 \geq 0\}} E^{S_1} \left[\mathbb{1}_{\{\tau_0 > (n-1)\}} \prod_{k=1}^{n-1} w_k(S_{n-k-1}) \right] \right] w_n(x) \\ &= b^n E^x \left[\mathbb{1}_{\{\tau_0 > n\}} \prod_{k=1}^{n} w_k(S_{n-k}) \right] \end{aligned}$$

which completes the proof. □

We state the key result of the section.

Proposition 3.3. *For any $x \geq 0$,*

$$E^x \left[\prod_{k=1}^{n} w_k(S_{n-k}) \,\middle|\, \tau_0 > n \right] \text{ converges to a positive constant as } n \to \infty. \tag{3.4}$$

Furthermore, the limit does not depend on the value of x.

Suppose that Proposition 3.3 holds. Let us see how it implies Theorem 1.1.

Proof of Theorem 1.1. Lemma 3.2 says that

$$P^x(Z_n > 0) = \mathbb{1}_{\{x \geq 0\}} b^n E^x \left[\prod_{k=1}^{n} w_k(S_{n-k}),\, \tau_0 > n \right]$$

Proposition 3.3 implies that there exists a constant C independent of x such that

$$P^x(Z_n > 0) \underset{n \to \infty}{\sim} C P^x(\tau_0 > n).$$

We conclude by equation (1.4). □

The remainder of the section is devoted to the proof of Proposition 3.3. The basic idea of the proof goes back to Harris and Harris [48], but several important ingredients (such as stochastic calculus and path decomposition for Bessel bridges) are no longer available in the discrete setting. First, we derive the convergence in law of S_n conditionally on $\tau_0 > n$. Then, we prove the lower limit and the upper limit of Proposition 3.3 in two distinct subsections.

3.1 A convergence in distribution

We use the Kolmogorov's extension theorem to define the probability $\mathbb{Q}$ such that for any n,

$$d\mathbb{Q}|_{\mathcal{F}_n} \quad := \quad \frac{\mathrm{e}^{\nu(S_n - S_0)}}{\phi(\nu)^n} dP|_{\mathcal{F}_n}\,,$$

where $\mathcal{F}_n$ is the sigma-algebra generated by $S_0, S_1, \ldots, S_n$. Under $\mathbb{Q}$, the random walk S_n is centered, and σ^2 defined by $\phi''(\nu)/\phi(\nu)$ is the variance of S_1 under $\mathbb{Q}$. Moreover,

$$\tilde{V}(x) = \sum_{k\geq 0} \mathbb{Q}^0(S_k = I_k(S) > -x) = V_S(x)$$

with the notation of Section 2. We introduce for any $z \geq 0$,

$$\tilde{V}_-(z) := 1 + \sum_{k\geq 1} \gamma^{-k} E^0 \left[\mathrm{e}^{\nu S_k} \mathbb{I}_{\{S_k = \sup_{0\leq \ell \leq k} S_\ell \leq z\}} \right]. \tag{3.5}$$

Then, with the notation of Section 2.2, we have $\tilde{V}_-(z) = V_{\overline{S}}(z)$. Let C_+ and C_- be the positive constants such that

$$\begin{aligned} \mathbb{Q}^0(\tau_0 > n) \quad &\sim_{n\to\infty} \quad \frac{C_+}{\sqrt{n}}, \\ \mathbb{Q}^0(\max_{0\leq k\leq n} S_k < 0) \quad &\sim_{n\to\infty} \quad \frac{C_-}{\sqrt{n}}. \end{aligned}$$

and S^* be a random variable on $(0, +\infty)$ with distribution given by

$$P(S^* \in dz) = \frac{1}{\int_0^\infty \mathrm{e}^{-\nu u} \tilde{V}_-(u) du} \mathrm{e}^{-\nu z} \tilde{V}_-(z) dz\,.$$

Lemma 3.4. *For any $x \geq 0$, the random variable S_n under $P^x(\cdot \,|\, \tau_0 > n)$ converges in law to S^*.*

Remark 3.5. The case $x = 0$ can be found in [55].

Proof. We first show that the sequence is tight. By changing measure from P^x to $\mathbb{Q}^x$, we have for any $A > 0$,

$$P^x(S_n > A,\, \tau_0 > n) = \mathrm{e}^{\nu x} E^x_{\mathbb{Q}} \left[\mathrm{e}^{-\nu S_n},\, S_n > A,\, \tau_0 > n \right]. \tag{3.6}$$

We see that

$$E^x_{\mathbb{Q}} \left[\mathrm{e}^{-\nu S_n},\, S_n > n^{1/3},\, \tau_0 > n \right] \leq \mathrm{e}^{-\nu n^{1/3}}. \tag{3.7}$$

Moreover,

$$E^x_{\mathbb{Q}}\left[\mathrm{e}^{-\nu S_n},\, S_n > A,\, \tau_0 > n\right] \le \sum_{\ell=\lfloor A \rfloor}^{\lfloor n^{1/3} \rfloor} \mathrm{e}^{-\nu\ell}\mathbb{Q}^x(S_n \in [\ell, \ell+1),\, \tau_0 > n).$$

We use Lemma 2.3 with $d_n = n^{1/3}$, $\delta = 1$ to see that there exists $c_{15} > 0$ and n_1 such that for any $n \ge n_1$ and any $A > 0$,

$$\sum_{\ell=\lfloor A \rfloor}^{\lfloor n^{1/3} \rfloor} \mathrm{e}^{-\nu\ell}\mathbb{Q}^x(S_n \in [\ell, \ell+1),\, \tau_0 > n). \le c_{15}\tilde{V}(x)\mathrm{e}^{-\nu A}n^{-3/2}. \tag{3.8}$$

Equations (3.6), (3.7) and (3.8) imply that for n greater than some n_2, we have uniformly in $A > 0$,

$$P^x(S_n > A,\, \tau_0 > n) \le \mathrm{e}^{\nu x}\tilde{V}(x)2c_{15}\mathrm{e}^{-\nu A}n^{-3/2}.$$

By equation (1.4), we obtain that there exists n_3 such that for any $n \ge n_3$ and any $A > 0$,

$$P^x(S_n > A \,|\, \tau_0 > n) \le \mathrm{e}^{-\nu A}(3c_{15}/C_2)$$

which proves the tightness. We prove now that the the sequence converges to S^*. It is enough to check the convergence on the particular test functions, $f(h) = \mathrm{e}^{\nu h}1_{\{h \in [z, z+\delta)\}}$ for $z \ge 0$ and $\delta > 0$. We write

$$\begin{aligned} E^x[f(S_n),\, \tau_0 > n] &= \gamma^n \mathrm{e}^{\nu x} E^x_{\mathbb{Q}}[\mathrm{e}^{-\nu S_n} f(S_n),\, \tau_0 > n] \\ &= \gamma^n \mathrm{e}^{\nu x}\mathbb{Q}^x(h \in [z, z+\delta),\, \tau_0 > n). \end{aligned}$$

We deduce from Lemma 2.3 that

$$E^x[f(S_n),\, \tau_0 > n] \underset{n\to\infty}{\sim} \gamma^{-n}\frac{C_-C_+}{2\sigma\sqrt{2\pi}}\frac{\tilde{V}(x)}{n^{3/2}}\int_z^{z+\delta}\tilde{V}_-(u)du.$$

Equation (1.4) yields that

$$\begin{aligned} \lim_{n\to\infty} E^x[f(S_n) \,|\, \tau_0 > n] &= \gamma^{-n}\frac{C_-C_+}{2\sigma\sqrt{2\pi}}\frac{\tilde{V}(x)}{C_2 n^{3/2}}\int_z^{z+\delta}\tilde{V}_-(u)du \\ &=: c_{11}\int_z^{z+\delta}\tilde{V}_-(u)du. \end{aligned}$$

We can rewrite the limit as $c_{11}\int_0^\infty f(u)\mathrm{e}^{-\nu u}\tilde{V}_-(u)du$. The convergence in distribution follows. Since the limiting measure is a probability distribution, we have $c_{11}\int_0^\infty f(u)\mathrm{e}^{-\nu u}\tilde{V}_-(u)du = 1$, which completes the proof. $\square$

3.2 Upper bound in Proposition 3.3

We prove a simpler version of equation (3.4).

Lemma 3.6. *Let $K \in \mathbb{N}^*$. Then*

$$\lim_{n\to\infty} E^x \left[\prod_{k=1}^{K} w_k(S_{n-k}) \middle| \tau_0 > n \right] = a_K \tag{3.9}$$

where a_K is defined in (3.10).

Proof. Let for any integer $i \in [0, K]$, $\widetilde{S}_i := S_{n-K+i} - S_{n-K}$. We have

$$E^x \left[\prod_{k=1}^{K} w_k(S_{n-k}),\, \tau_0 > n \right] \quad = \quad E^x \left[\prod_{k=0}^{K-1} w_{K-k}(S_{n-K} + \widetilde{S}_k),\, \tau_0 > n \right]$$

Since $\widetilde{S}$ is independent of S_{n-K}, we can write

$$E^x \left[\prod_{k=0}^{K-1} w_{K-k}(S_{n-K} + \widetilde{S}_k),\, \tau_0 > n \right] \quad = \quad E^x \left[f_K(S_{n-K}),\, \tau_0 > n - K \right]$$

with $f_K(z) := E^z \left[\mathbb{1}_{\{\tau_0 > K\}} \prod_{k=0}^{K-1} w_{K-k}(S_k) \right]$. By Lemma 3.4, we know that S_n, conditioned on $\tau_0 > n$, converges in distribution to S^*. Then by the continuous mapping theorem, $E^x \left[f_K(S_{n-K}) \,|\, \tau_0 > n - K \right]$ converges to $E\left[f_K(S^*) \right]$. We are allowed to make use of the continuous mapping theorem because S^* has a density and f_K has at most countably many points of discontinuity (indeed, these are the points from which the random walk has a positive probability to reach the origin in at most K steps and stay positive before. They are related to the atoms of the law of X which are at most countably many). Then,

$$E^x \left[\prod_{k=1}^{K} w_k(S_{n-k}) \middle| \tau_0 > n \right] = \frac{P^x(\tau_0 > n - K)}{P^x(\tau_0 > n)} E^x \left[f_K(S_{n-K}) \,|\, \tau_0 > n - K \right],$$

which tends to

$$a_K := \gamma^{-K} E\left[f_K(S^*) \right] \tag{3.10}$$

by equation (1.4). □

We deduce the following upper bound.

Corollary 3.7. *We have*

$$\limsup_{n\to\infty} E^x\left[\prod_{k=1}^{n}(1-w_k(S_{n-k}))\,\middle|\,\tau_0>n\right]\le\inf_{K\ge1}a_K\,.$$

Proof. We observe that for any $n\ge K\ge 1$,

$$E^x\left[\prod_{k=1}^{n}w_k(S_{n-k})\,\middle|\,\tau_0>n\right]\le E^x\left[\prod_{k=1}^{K}w_k(S_{n-k})\,\middle|\,\tau_0>n\right]\,.$$

By Lemma 3.6, it implies that

$$\limsup_{n\to\infty} E^x\left[\prod_{k=1}^{n}w_k(S_{n-k})\,\middle|\,\tau_0>n\right]\le a_K.$$

Take the infimum over $K\ge1$ to complete the proof. □

3.3 Lower bound in Proposition 3.3

We show that $\inf_{K\ge1}a_K$ is also a lower bound.

Lemma 3.8. *We have*

$$\liminf_{n\to\infty} E^x\left[\prod_{k=1}^{n}w_k(S_{n-k})\,\middle|\,\tau_0>n\right]\ge\inf_{K\ge1}a_K \tag{3.11}$$

and $\inf_{K\ge1}a_K>0$.

Proof. Let $\varepsilon>0$ and $\eta>0$. We first prove that there exists K large enough such that

$$E^x\left[\prod_{k=1}^{n}w_k(S_{n-k})\,\middle|\,\tau_0>n\right]\ge(1-\varepsilon)E^x\left[\prod_{k=1}^{K}w_k(S_{n-k})\,\middle|\,\tau_0>n\right]-\eta\,. \tag{3.12}$$

Since $w_k\le1$, we have, for any $K\le n$,

$$\begin{aligned}&(1-\varepsilon)E^x\left[\prod_{k=1}^{K}w_k(S_{n-k})\,\middle|\,\tau_0>n\right]\\ \le\;& E^x\left[\prod_{k=1}^{n}w_k(S_{n-k})\,\middle|\,\tau_0>n\right]+P^x\left(\prod_{k=K+1}^{n}w_k(S_{n-k})<1-\varepsilon\,\middle|\,\tau_0>n\right).\end{aligned}$$

Therefore, we need to show that

$$P^x\left(\prod_{k=K+1}^{n}w_k(S_{n-k})<1-\varepsilon\,\middle|\,\tau_0>n\right)\le\eta$$

when K is large. We split it into three parts, by observing that

$$\begin{aligned}
&P^x\left(\prod_{k=K+1}^{n} w_k(S_{n-k}) < 1-\varepsilon \,\middle|\, \tau_0 > n\right) \\
&\leq P^x(A_1 \,|\, \tau_0 > n) + P^x(A_2 \,|\, \tau_0 > n) + P^x\left(\prod_{k=K+1}^{n} w_k(S_{n-k}) < 1-\varepsilon,\, A_3 \,\middle|\, \tau_0 > n\right)
\end{aligned}$$

with

$$\begin{aligned}
A_1 &:= \{S_n \geq M\}\,, \\
A_2 &:= \left\{\exists\, K+1 \leq k \leq n \text{ such that } S_{n-k} \geq M + k^{2/3},\, S_n \leq M\right\}, \\
A_3 &:= A_1^c \cap A_2^c.
\end{aligned}$$

By Lemma 3.4, the sequence $(S_n)_{n\geq 0}$ conditionally on $\{\tau_0 > n\}$ converges in distribution to S^*. Therefore there exists $M = M(\eta)$ such that $P^x\left(A_1 \,\middle|\, \tau_0 > n\right) < \eta/2$ for n large enough. Let us consider A_2.

$$P^x\left(A_2 \,\middle|\, \tau_0 > n\right) \leq \sum_{k=K+1}^{n} P^x\left(S_{n-k} \geq M + k^{2/3},\, S_n \leq M \,\middle|\, \tau_0 > n\right).$$

We use Lemma 6.2 (see Appendix) with $\alpha = 2/3$. There exist some constants $c_M > 0$ and $b > 0$ such that, for any $k \in [\![0, n]\!]$,

$$P^x\left(S_{n-k} \geq M + k^{2/3},\, S_n \leq M \,\middle|\, \tau_0 > n\right) \leq c_M \mathrm{e}^{-k^b}.$$

It implies that

$$P^x\left(A_2 \,\middle|\, \tau_0 > n\right) \leq c_M \sum_{k\geq K+1} \mathrm{e}^{-k^b} \leq \eta/2$$

for $K = K(\eta)$ large enough. Therefore, we found $M(\eta)$ and $K(\eta)$ such that

$$P^x(A_1) + P^x(A_2) \leq \eta.$$

It remains to bound the probability of having A_3 and $\{\prod_{k=K+1}^{n} w_k(S_{n-k}) < 1-\varepsilon\}$. From (3.2), we know that $1 - w_k(x) \leq c_3 u_k(x)$. Furthermore,

$$u_k(x) = P^x(Z_k > 0) \leq E^x[Z_k] = b^k P^x(\tau_0 > k)\,.$$

We observe that $P^x(\tau_0 > k) \leq P^x(S_k \geq 0)$ and we use the Cramér's bound $P^x(S_k \geq 0) \leq E^0\left[\mathrm{e}^{\nu(x+S_k)}\right] = \mathrm{e}^{\nu x}\phi(\nu)^k$ to get that

$$1 - w_k(x) \leq c_3 \mathrm{e}^{\nu x}\phi(\nu)^k\,. \tag{3.13}$$

On the event A_3, we have

$$\begin{aligned}\prod_{k=K+1}^{n} w_k(S_{n-k}) &\geq \prod_{k=K+1}^{n}\left(1 - c_3\mathrm{e}^{\nu(M+k^{2/3})}\phi(\nu)^k\right)\\ &\geq \prod_{k=K+1}^{\infty}\left(1 - c_3\mathrm{e}^{\nu(M+k^{2/3})}\phi(\nu)^k\right) =: F(K)\,.\end{aligned} \tag{3.14}$$

Since $\lim_{K\to\infty} F(K) = 0$, we can choose $K = K(\varepsilon)$ large enough to have $F(K) > 1-\varepsilon$. Hence we get for K large enough,

$$P^x\left(\prod_{k=K+1}^{n} w_k(S_{n-k}) < 1-\varepsilon,\, A_3 \,\middle|\, \tau_0 > n\right) = 0.$$

This proves (3.12). In particular, taking the limit $n \to \infty$, then $\inf_{K\geq 1}$, we obtain

$$\liminf_{n\to\infty} E^x\left[\prod_{k=1}^{n} w_k(S_{n-k}) \,\middle|\, \tau_0 > n\right] \geq (1-\varepsilon)\inf_{K\geq 1} a_K - \eta\,.$$

We take $\eta \to 0$ and $\varepsilon \to 0$ to complete the proof of (3.11). Combined with Corollary 3.7, it implies that

$$\lim_{n\to\infty} E^x\left[\prod_{k=1}^{n} w_k(S_{n-k}) \,\middle|\, \tau_0 > n\right] = \inf_{K\geq 1} a_K. \tag{3.15}$$

We show now that $\inf_{K\geq 1} a_K > 0$. Let $\mu > 0$ and $\eta > 0$. We write as before

$$\begin{aligned}&P^x\left(\prod_{k=1}^{n} w_k(S_{n-k}) < \mu \,\middle|\, \tau_0 > n\right)\\ &\qquad\leq P^x(A_1 \,|\, \tau_0 > n) + P^x(A_2 \,|\, \tau_0 > n) + P^x\left(\prod_{k=0}^{n} w_k(S_{n-k}) < \mu,\, A_3 \,|\, \tau_0 > n\right).\end{aligned}$$

We already showed that there exists $M = M(\eta)$ and $K = K(\eta)$ such that for n large enough

$$P^x(A_1 \,|\, \tau_0 > n) + P^x(A_2 \,|\, \tau_0 > n) \leq \eta. \tag{3.16}$$

We recall also that on the event A_3, we have

$$\prod_{k=K(\eta)+1}^{n} w_k(S_{n-k}) \geq F(K)$$

where $F(K)$ is defined in (3.14). We take care of choosing K big enough to have $F(K) > 0$. We emphasize that K does not depend so far on the value of μ. We have for $n \geq K$,

$$P^x\left(\prod_{k=0}^{n} w_k(S_{n-k}) < \mu, A_3 \,\middle|\, \tau_0 > n\right) \leq P^x\left(\prod_{k=0}^{K} w_k(S_{n-k}) < \mu/F(K) \,\middle|\, \tau_0 > n\right). \tag{3.17}$$

By the Markov property,

$$P^x\left(\prod_{k=0}^{K} w_k(S_{n-k}) < \mu/F(K),\, \tau_0 > n\right) = E^x\left[g_{K,\mu}(S_{n-K}),\, \tau_0 > n-K\right]$$

where $g_{K,\mu}(z) := P^z\left(\prod_{k=0}^{K} w_k(S_{K-k}) < \mu/F(K),\, \tau_0 > K\right)$. By Lemma 3.4, using the continuous mapping theorem yields that $E^z[g_{K,\mu}(S_{n-K}) \,|\, \tau_0 > n-K]$ tends to $\gamma^{-K}E[g_{K,\mu}(S^*)]$. Again, we used the fact that the function $g_{K,\mu}$ has only countable many discontinuities. Let $\mu > 0$ small enough to have $\gamma^{-K}E[g_{K,\mu}(S^*)] \leq \eta$. By equations (3.16) and (3.17), we have for $n \geq K$,

$$\begin{aligned} E^x\left[\prod_{k=0}^{n} w_k(S_{n-k}) \,\middle|\, \tau_0 > n\right] &\geq \mu P^x\left(\prod_{k=0}^{n} w_k(S_{n-k}) > \mu \,\middle|\, \tau_0 > n\right) \\ &\geq \mu\left(1 - \eta - P^x\left(\prod_{k=0}^{K} w_k(S_{n-k}) < \mu/F(K) \,\middle|\, \tau_0 > n\right)\right). \end{aligned}$$

We take the limit $n \to \infty$. By (3.15), the LHS goes to $\inf_{K\geq 1} a_K$. Therefore,

$$\inf_{K\geq 1} a_K \geq \mu(1-2\eta) > 0$$

if η is taken strictly smaller than $1/2$. □

4 Mogul'skii's estimate and corollaries

Before the proof of Theorem 1.2, we mention a small deviations result of Mogul'skii, which gives the probability for a random walk to stay between two curves.

For any L, $\widetilde{L} \in \mathcal{F}[0,1]$, we write $L < \widetilde{L}$ when $\forall t \in [0,1], L(t) < \widetilde{L}(t)$ and $L \leq \widetilde{L}$ when $\forall t \in [0,1], L(t) \leq \widetilde{L}(t)$. If $n \geq 1$, we write $L <_n \widetilde{L}$ when $\forall 1 \leq k \leq n$, $L\left(\frac{k}{n}\right) < \widetilde{L}\left(\frac{k}{n}\right)$ and $L \leq_n \widetilde{L}$ when $\forall 1 \leq k \leq n$, $L\left(\frac{k}{n}\right) \leq \widetilde{L}\left(\frac{k}{n}\right)$.

Theorem 4.1 (Mogul'skii)**.** *Let $\xi_1,\, \xi_2,\, \ldots$ be i.i.d. random variables such that $E[\xi_1] = 0$ and $\sigma^2 := E[\xi_1^2] < \infty$. Let $(x_n,\, n \geq 0)$ be a sequence of positive numbers such that*

$$\begin{aligned} \lim_{n\to\infty} x_n &= +\infty\,, \\ \lim_{n\to\infty} \frac{x_n}{\sqrt{n}} &= 0\,. \end{aligned}$$

Define for any $n \geq 0$

$$S_n := S_0 + \xi_1 + \xi_2 + \ldots + \xi_n,$$

where $S_0 = z$ almost surely under the probability P^z ($z \in \mathbb{R}$).
When $z = 0$, write $P := P^0$ and define, for any $t \in [0,1]$,

$$s_n(t) := \frac{S_{\lfloor tn \rfloor}}{x_n} = \frac{\xi_1 + \xi_2 + \ldots + \xi_k}{x_n} \qquad \text{for} \qquad k/n \leq t < (k+1)/n\,.$$

Then, for any $L_1,\, L_2 \in \mathcal{C}[0,1]$, with $L_1(t) < L_2(t)$, we have, as $n \to \infty$,

$$\log\left(P(L_1 < s_n < L_2)\right) \sim -C_{L_1,L_2} n x_n^{-2},$$

where

$$C_{L_1,L_2} := \frac{\pi^2\sigma^2}{2} \int_0^1 \frac{\mathrm{d}t}{\left(L_2(t) - L_1(t)\right)^2}\,.$$

We keep the notations and assumptions of Theorem 4.1 throughout this section. For the proof, we refer to [91].

Lemma 4.2. *Set $L_1,\, L_2 \in \mathcal{C}[0,1]$, with $L_1 < L_2$ and $L_1(0) < 0 < L_2(0)$. For any sequences $(L_1^n)_n$ and $(L_2^n)_n$ of $\mathcal{F}[0,1]$ such that $\|L_1^n - L_1\|_\infty \to 0$ and $\|L_2^n - L_2\|_\infty \to 0$ as $n \to \infty$, we have*

$$\log\left(P(L_1^n < s_n < L_2^n)\right) \sim -n x_n^{-2} C_{L_1,L_2}.$$

Remark 4.3. In the conclusion of Lemma 4.2, we can replace the strict inequalities

$$L_1^n < s_n < L_2^n$$

by weak ones (or take one strict and the other weak) and obtain the same estimate by exactly the same argument. One can easily check that this also applies to the other results of this section since we deduce them from Lemma 4.2.

Proof of Lemma 4.2. Let $0 < \varepsilon < \frac{1}{2}\min_{[0,1]}(L_2 - L_1)$. We can choose $N \geq 1$ such that

$$\forall j \in \{1,2\}, \forall n \geq N, \max_{[0,1]} |L_j^n - L_j| < \varepsilon.$$

Then, for any $n \geq N$, we have

$$\{L_1 + \varepsilon < s_n < L_2 - \varepsilon\} \subset \{L_1^n < s_n < L_2^n\} \subset \{L_1 - \varepsilon < s_n < L_2 + \varepsilon\}.$$

Using the corresponding inequalities for probabilities and applying Theorem 4.1, we get

$$-C_{L_1+\varepsilon, L_2-\varepsilon} \leq \liminf_{n\to\infty} \frac{x_n^2}{n} \log P(L_1^n < s_n < L_2^n) \leq \limsup_{n\to\infty} \frac{x_n^2}{n} \log P(L_1^n < s_n < L_2^n) \leq -C_{L_1-\varepsilon, L_2+\varepsilon}.$$

We make $\varepsilon \to 0$. Then $C_{L_1+\varepsilon, L_2-\varepsilon} \to C_{L_1,L_2}$ and $C_{L_1-\varepsilon, L_2+\varepsilon} \to C_{L_1,L_2}$, and the lemma is proved. □

Lemma 4.4. *Set L_1, $L_2 \in \mathcal{C}[0,1]$, with $L_1 < L_2$ and $L_1(0) < 0 < L_2(0)$. For any sequences $(L_1^n)_n$ and $(L_2^n)_n$ of $\mathcal{F}[0,1]$ such that $\|L_1^n - L_1\|_\infty \to 0$ and $\|L_2^n - L_2\|_\infty \to 0$ as $n \to \infty$, we have*

$$\log\left(P(L_1^n <_n s_n <_n L_2^n)\right) \sim -n x_n^{-2} C_{L_1,L_2}.$$

Proof. We introduce the sequences of piecewise constant interpolation functions $(\widetilde{L}_1^n)_n$ and $(\widetilde{L}_2^n)_n$ defined by :

$$\forall j \in \{1,2\}, \forall n \geq 1, \forall 0 \leq t \leq 1, \; \widetilde{L}_j^n(t) := L_j^n\left(\frac{\lfloor tn \rfloor}{n}\right).$$

We notice that

$$\forall n \geq 1, \{L_1^n <_n s_n <_n L_2^n\} = \{\widetilde{L}_1^n < s_n < \widetilde{L}_2^n\}.$$

Before concluding by applying Lemma 4.2 with $\widetilde{L}_j^n$ playing the role of L_j^n, we have to check that these sequences converge uniformly to L_j as $n \to \infty$ for $j = 1,2$. Set $\varepsilon > 0$. We can choose $N \geq 1$ such that for $j = 1,2$ for any $n \geq N$, we have $\|L_j^n - L_j\|_\infty < \varepsilon$.
We can also choose $\eta > 0$ such that for $j = 1,2$,

$$\forall t, t' \in [0,1], (|t - t'| \leq \eta \Rightarrow |L_j(t) - L_j(t')| < \varepsilon).$$

Let $t \in [0,1]$ and $n \geq \frac{1}{\eta}$. Noticing that $|\frac{\lfloor tn \rfloor}{n} - t| \leq \frac{1}{n}$, we get for $j = 1,2$

$$\forall n \geq \max(N, \frac{1}{\eta}), |\widetilde{L}_j^n(t) - L_j(t)| \leq |L_j^n\left(\frac{\lfloor tn \rfloor}{n}\right) - L_j\left(\frac{\lfloor tn \rfloor}{n}\right)| + |L_j\left(\frac{\lfloor tn \rfloor}{n}\right) - L_j(t)| \leq 2\varepsilon.$$

Then $\|\widetilde{L}_j^n - L_j^n\|_\infty \to 0$ as $n \to \infty$ and we can apply Lemma 4.2, which proves Lemma 4.4. □

From now on, we set :

$$\forall n \geq 1,\, x_n := n^{1/3}. \tag{4.1}$$

Lemma 4.5. *Set L_1, $L_2 \in \mathcal{C}[0,1]$, with $L_1 < L_2$ and $L_1(0) < 0 < L_2(0)$. Let $(L_1^n)_n$ and $(L_2^n)_n$ be sequences of $\mathcal{F}[0,1]$ such that $\|L_1^n - L_1\|_\infty \to 0$ and $\|L_2^n - L_2\|_\infty \to 0$ as $n \to \infty$. Let β^* and γ^* be positive real numbers such that $0 \leq \beta^* < \gamma^* \leq 1$. Let $u^*, v^* \in \mathbb{R}$ such that $L_1(\beta^*) \leq u^* < v^* \leq L_2(\beta^*)$. Let $(u_n)_n$ and $(v_n)_n$ be sequences of reals satisfying*

$$\frac{u_n}{x_n} \to u^*, \quad \frac{v_n}{n} \to v^*, \quad L_1^n(\beta(n))x_n \leq u_n \leq v_n \leq L_2^n(\beta(n))x_n \quad \forall n \geq 1.$$

Let $(\beta(n))_n$ and $(\gamma(n))_n$ be sequences of reals satisfying

$$\frac{\beta(n)}{n} \to \beta^*, \quad \frac{\gamma(n)}{n} \to \gamma^*, \quad 1 \leq \beta(n) < \gamma(n) \leq n, \forall n \geq 1.$$

We also assume that :

$$\exists M \in \mathbb{N}^*, \forall m \in \mathbb{N}^*, \#\{n : (\gamma - \beta)(n) = m\} \leq M.$$

It is easy to see that the last condition holds if the sequence $(\gamma(n) - n\gamma^* - \beta(n) + n\beta^*)_n$ is bounded. Then

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \left(\sup_{u_n \leq z \leq v_n} P^z \left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-\beta(n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall \beta(n) < k \leq \gamma(n) \right)\right) \leq -C^{\beta^*,\gamma^*}_{L_1 - v^*, L_2 - u^*},$$

where, for any functions continuous L_1 and $L_2 : [0,1] \mapsto \mathbb{R}$ such that $L_1 \leq L_2$,

$$C^{\beta^*,\gamma^*}_{L_1,L_2} := \frac{\pi^2\sigma^2}{2} \int_{\beta^*}^{\gamma^*} \frac{\mathrm{d}t}{(L_2(t) - L_1(t))^2}.$$

Proof. Write $m := \gamma(n) - \beta(n)$. Notice that

$$\begin{aligned} &\left\{\forall z \in [u_n, v_n], \forall k \leq m, x_n L_1^n\left(\frac{\beta(n)+k}{n}\right) < z + S_k < x_n L_2^n\left(\frac{\beta(n)+k}{n}\right)\right\} \\ \subset\; &\left\{\forall k \leq m, x_n\left(L_1^n\left(\frac{\beta(n)+k}{n}\right) - v_n\right) < S_k < x_n\left(L_2^n\left(\frac{\beta(n)+k}{n}\right) - u_n\right)\right\}. \end{aligned} \tag{4.2}$$

Let $A = \{m \in \mathbb{N}^* : \exists n \in \mathbb{N}^*, \gamma(n) - \beta(n) = m\}$.
By hypothesis, each $m \in A$ is attained by $\gamma - \beta$ for at most M different values of n, hence we can define a surjection $\varphi : \{1, 2, \ldots, M\} \times A \mapsto \mathbb{N}^*$ such that for any $1 \leq l \leq M$, $\varphi(l,m) =: n$ satisfies $\gamma(n) - \beta(n) = m$.

For each $1 \le l \le M$, we define

$$\widetilde{L}_1^m(t) = \frac{L_1^n\left(\frac{(1-t)\beta(n)+t\gamma(n)}{n}\right)x_n - v_n}{x_m}, \quad \widetilde{L}_2^m(t) = \frac{L_2^n\left(\frac{(1-t)\beta(n)+t\gamma(n)}{n}\right)x_n - u_n}{x_m},$$

where $n := \varphi(l,m)$. Using (2.2) and $n \sim (\gamma^* - \beta^*)^{-1}m$, we obtain $x_n \sim x_m(\gamma^* - \beta^*)^{-1/3}$. Consequently these sequences of functions satisfy, as $m \to \infty$ (and so $n := \varphi(l,m) \to \infty$) :

$$\begin{aligned}
\widetilde{L}_1^m(t) \to \widetilde{L}_1(t) &:= (\gamma^* - \beta^*)^{-1/3}\left[L_1\left((1-t)\beta^* + t\gamma^*\right) - v^*\right], \\
\widetilde{L}_2^m(t) \to \widetilde{L}_2(t) &:= (\gamma^* - \beta^*)^{-1/3}\left[L_2\left((1-t)\beta^* + t\gamma^*\right) - u^*\right].
\end{aligned}$$

For each $1 \le l \le M$, we apply Lemma 4.4 with $\widetilde{L}_1^m$ and $\widetilde{L}_2^m$ to the probability of the event in the right-hand side of (2.3), we obtain, for the event on the left-hand side, as $m \to \infty$

$$\begin{aligned}
& P\left(\forall z \in [u_n, v_n], \forall k \le m, x_n L_1^n\left(\frac{\beta(n)+k}{n}\right) < z + S_k < x_n L_2^n\left(\frac{\beta(n)+k}{n}\right)\right) \\
\le\ & \exp\left(-(1+o(1))m^{1/3}\frac{\pi^2\sigma^2}{2}\int_0^1 \frac{\mathrm{d}t}{\left(\widetilde{L}_2(t) - \widetilde{L}_1(t)\right)^2}\right) \\
=\ & \exp\left(-(1+o(1))n^{1/3}\frac{\pi^2\sigma^2}{2}\int_{\beta^*}^{\alpha^*} \frac{\mathrm{d}t}{\left(L_2(t) - u^* - L_1(t) + v^*\right)^2}\right) \\
=\ & \exp\left(-(1+o(1))n^{1/3}C_{L_1-v^*,L_2-u^*}^{\beta^*,\gamma^*}\right).
\end{aligned}$$

This bound holds with n running along the M subsequences $\varphi(l,m)_m$, $1 \le l \le M$, which together cover all the values $n \in \mathbb{N}^*$, and thus Lemma 4.5 is proved. □

Lemma 4.6. *Set L_1, $L_2 \in \mathcal{C}[0,1]$, with $L_1 < L_2$ and $L_1(0) < 0 < L_2(0)$. Let $(L_1^n)_n$ and $(L_2^n)_n$ be sequences of $\mathcal{F}[0,1]$ such that $\|L_1^n - L_1\|_\infty \to 0$ and $\|L_2^n - L_2\|_\infty \to 0$ as $n \to \infty$. Let β^* and γ^* be positive real numbers such that $0 \le \beta^* < \gamma^* \le 1$. Let $(\beta(n))_n$ and $(\gamma(n))_n$ be sequences of reals satisfying*

$$\frac{\beta(n)}{n} \to \beta^*, \quad \frac{\gamma(n)}{n} \to \gamma^*, \quad 1 \le \beta(n) < \gamma(n) \le n \quad \forall n \ge 1.$$

We also assume :

$$\exists M \in \mathbb{N}^*, \forall m \in \mathbb{N}^*, \#\{n : (\gamma - \beta)(n) = m\} \le M.$$

It is easy to see that the last condition holds if the sequence $(\gamma(n) - n\gamma^* - \beta(n) + n\beta^*)_n$ *is bounded. Then*

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \left(\sup_z P^z \left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-\beta(n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall \beta(n) < k \le \gamma(n) \right) \right) \\ \le\ & -C^{\beta^*,\gamma^*}_{L_1,L_2}, \end{aligned}$$

where the $\sup_z$ *is over the* $z \in \mathbb{R}$ *such that* $x_n L_1^n(\beta(n)) \le z \le x_n L_2^n(\beta(n))$.

Proof. Let $\varepsilon > 0$. Let N be an integer such that $N\varepsilon > L_2(\beta^*) - L_1(\beta^*)$. We define for $j = 0, 1, \ldots, N$,

$$u_n^j := x_n \frac{L_1^n(\beta^*)(N-j) + L_2^n(\beta^*)j}{N}.$$

With obvious notation, we have

$$\sup_{x_n L_1^n(\beta(n)) \le z \le x_n L_2^n(\beta(n))} p(z,n) = \max_{0\le j\le N-1} \sup_{u_n^j \le z \le v_n^j} p(z,n).$$

We apply Lemma 4.5 N times, with $u_n = u_n^j$ and $v_n = u_n^{j+1}$, $J = 0, 1, \ldots, N-1$ and get by the preceding equation :

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \left(\sup_z P^z \left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-\beta(n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall \beta(n) < k \le \gamma(n) \right) \right) \\ (4.3) \le\ & -C^{\beta^*,\gamma^*}_{L_1,L_2+\frac{L_2(\beta^*)-L_1(\beta^*)}{N}} \le -C^{\beta^*,\gamma^*}_{L_1,L_2+\varepsilon}, \end{aligned}$$

where the $\sup_z$ is over the $z \in \mathbb{R}$ such that $x_n L_1^n(\beta(n)) \le z \le x_n L_2^n(\beta(n))$.
In the computation of (2.4), we used the fact, obvious from its definition, that $C^{\beta^*,\gamma^*}_{L_1,L_2}$ only depends on $L_2 - L_1$, β^* and γ^*, which implies

$$\forall 0 \le j \le N-1, C^{\beta^*,\gamma^*}_{L_1 - \frac{L_1^n(\beta^*)(N-j-1)+L_2^n(\beta^*)(j+1)}{N}, L_2 - \frac{L_1^n(\beta^*)(N-j)+L_2^n(\beta^*)j}{N}} = C^{\beta^*,\gamma^*}_{L_1,L_2+\frac{L_2(\beta^*)-L_1(\beta^*)}{N}}.$$

We make $\varepsilon \to 0$, then $C^{\beta^*,\gamma^*}_{L_1,L_2+\varepsilon} \to C^{\beta^*,\gamma^*}_{L_1,L_2}$ and we conclude using the bound (2.4). □

Proposition 4.7. *Set* L_1, $L_2 \in \mathcal{C}[0,1]$, *with* $L_1 < L_2$ *and* $L_1(0) < 0 < L_2(0)$. *Let* $(L_1^n)_n$ *and* $(L_2^n)_n$ *be sequences of* $\mathcal{F}[0,1]$ *such that* $\|L_1^n - L_1\|_\infty \to 0$ *and* $\|L_2^n - L_2\|_\infty \to 0$ *as* $n \to \infty$. *We assume* B *and* C *are mappings* $[0,1] \times \mathbb{N}^* \mapsto \mathbb{N}^*$, *nondecreasing in the first component and such that, for any* $\alpha \in [0,1]$, *the sequences* $(B(\alpha,n) - \alpha n)_n$ *and* $(C(\alpha,n) - \alpha n)_n$ *are bounded.*

We know from Lemma 4.6 that for any $0 \leq \beta^* < \gamma^* \leq 1$, we have

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log\left(\sup_z P^z\left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-B(\beta^*,n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(\beta^*,n) < k \leq C(\gamma^*,n)\right)\right)$$
$$\leq -C_{L_1,L_2}^{\beta^*,\gamma^*},$$

where the $\sup_z$ is over the $z \in \mathbb{R}$ such that $x_n L_1^n(B(\beta^*,n)) \leq z \leq x_n L_2^n(B(\beta^*,n))$.

The claim of the present proposition is that this estimate holds uniformly in β^* and γ^* such that $0 \leq \beta^* < \gamma^* \leq 1$.

Proof. Let $\varepsilon > 0$. Let N be an integer such that

$$\forall 0 \leq \beta^* \leq \gamma^* \leq 1, (\gamma^* - \beta^* < \frac{1}{N} \Rightarrow C_{L_1,L_2}^{\beta^*,\gamma^*} < \varepsilon).$$

We apply Lemma 4.6 $N(N+1)/2$ times with

$$\beta^* = \frac{b}{N}, \quad \gamma^* = \frac{c}{N}, \quad 0 \leq b < c \leq N.$$

Then for n big enough, and any integers b and c such that $0 < b \leq c < N$, we have

$$\frac{1}{n^{1/3}} \log\left(\sup_z P^z\left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-B(\frac{b}{N},n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(\frac{b}{N},n) < k \leq C(\frac{c}{N},n)\right)\right)$$
$$\leq -C_{L_1,L_2}^{\frac{b}{N},\frac{c}{N}} + \varepsilon,$$

where the $\sup_z$ is over the $z \in \mathbb{R}$ such that $x_n L_1^n(B(\beta^*,n)) \leq z \leq x_n L_2^n(B(\beta^*,n))$.
For any $0 \leq \beta^* \leq \gamma^* \leq 1$, we can find $1 \leq b \leq N$ and $0 \leq c \leq N-1$ such that :

$$\frac{b-1}{N} \leq \beta^* \leq \frac{b}{N}, \quad \frac{c}{N}\gamma^* \leq \frac{c+1}{N}.$$

If $b \leq c$, then

$$\begin{aligned}
&\frac{1}{n^{1/3}} \log\left(\sup_z P\left(L_1^n\left(\frac{k}{n}\right) < \frac{z+S_{k-B(\beta^*,n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(\beta^*,n) < k \leq C(\gamma^*,n)\right)\right) \\
\leq\ & \frac{1}{n^{1/3}} \log\left(\sup_z P\left(L_1^n\left(\frac{k}{n}\right) < \frac{z+S_{k-B(\frac{b}{N},n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(\frac{b}{N},n) < k \leq -C(\frac{c}{N},n)\right)\right) \\
\leq\ & -C_{L_1,L_2}^{\frac{b}{N},\frac{c}{N}} + \varepsilon \\
\leq\ & -C_{L_1,L_2}^{\beta^*,\gamma^*} + 3\varepsilon,
\end{aligned}$$

where the $\sup_z$ is over the $z \in \mathbb{R}$ such that $x_n L_1^n(B(\beta^*, n)) \leq z \leq x_n L_2^n(B(\beta^*, n))$.

Else $b = c+1$, $\gamma^* - \beta^* \leq 1/N$, hence $C_{L_1,L_2}^{\beta^*,\gamma^*} < \varepsilon$. This case is easier :

$$\begin{aligned}
&\frac{1}{n^{1/3}} \log \left(\sup_z P^z \left(L_1^n \left(\frac{k}{n} \right) < \frac{S_{k-B(\beta^*,n)}}{x_n} < L_2^n \left(\frac{k}{n} \right), \forall B(\beta^*, n) < k \leq C(\gamma^*, n) \right) \right) \\
\leq \;\; & 0 \leq -C_{L_1,L_2}^{\beta^*,\gamma^*} + \varepsilon.
\end{aligned}$$

This ends the proof of the proposition. □

Remark 4.8. The upper bound above is sharp and may be replaced by an equivalence. Keeping the above notations and hypothesis, we have, for any $\varepsilon > 0$ small enough,

$$\log \left(\inf_z P^z \left(L_1^n \left(\frac{k}{n} \right) < \frac{S_{k-\beta(n)}}{x_n} < L_2^n \left(\frac{k}{n} \right), \forall \beta(n) < k \leq \gamma(n) \right) \right) \sim -n x_n^{-2} C_{L_1,L_2}^{\beta^*,\gamma^*},$$

where the $\inf_z$ is over the $z \in \mathbb{R}$ such that $x_n \left(L_1^n(B(\beta^*, n)) + \varepsilon \right) < z < x_n \left(L_2^n(B(\beta^*, n)) - \varepsilon \right)$. The proof of this result is very similar to the upperbound, but since it is not useful here, we omit it.

The following lemma, the proof of which is trivial and omitted, makes Proposition 4.7 easier to apply.

Lemma 4.9. *Under the assumptions of Proposition 4.7,* $C_{L_1,L_2}^{\frac{\beta(n)}{n},\frac{\gamma^*}{n}} \to C_{L_1,L_2}^{\beta^*,\gamma^*}$, *uniformly in* $0 \leq \beta^* \leq \gamma^* \leq 1$ *as* $n \to \infty$.

Remark 4.10. For the upper bounds in the preceding lemmas and in Proposition 4.7, we can release the hypothesis that $L_1 < L_2$ and $L_1(0) < 0 < L_2(0)$. The following argument extends the upper bound to functions satisfying $L_1 \leq L_2$, $C_{L_1,L_2} < \infty$ and $L_1(0) \leq 0 \leq L_2(0)$ (instead of the stronger conditions $L_1 < L_2$ and $L_1(0) < 0 < L_2(0)$) :

Let $\varepsilon > 0$. Notice that the probability that s_n stay between L_1^n and L_2^n is less than the probability that s_n stays between $\widetilde{L}_1^n := L_1^n - \varepsilon$ and $\widetilde{L}_2^n := L_2^n + \varepsilon$. Hence we may apply for example Proposition 4.7 with $\widetilde{L}_1^n$ and $\widetilde{L}_2^n$ and obtain a uniform upper bound $\exp\left(-n^{1/3} C_{\widetilde{L}_1,\widetilde{L}_2}^{\beta^*,\gamma^*}(1+o(1))\right)$. Now let $\varepsilon \to 0$, $C_{\widetilde{L}_1,\widetilde{L}_2}^{\beta^*,\gamma^*} \to C_{L_1,L_2}^{\beta^*,\gamma^*}$ uniformly in β^* and γ^*.

5 The Critical Case

Remind that $\mathbb{Q}^x$, defined by

$$d\mathbb{Q}|_{\mathcal{F}_n} \;\; := \;\; \frac{\mathrm{e}^{\nu S_n}}{\phi(\nu)^n} dP|_{\mathcal{F}_n} \, ,$$

is a probability under which S_n is centered. Let us define $\sigma^2 := E_{\mathbb{Q}}[X_1^2]$.

We turn to the proof of Theorem 1.2. Let $(f_k,\, k \geq 0)$ be for the time being any sequence of positive reals. We will precise the value of f_k later on. For any $|u| = k$, we say that $u \in \mathcal{S}$ if for any $\ell \leq k$, the ancestor u_ℓ of u at generation ℓ verifies $0 < S(u_\ell) \leq f_\ell$. We introduce

$$Z_n(\mathcal{S}) := \sum_{|x|=n} \mathbb{1}_{\{x\in\mathcal{S}\}}\,.$$

In words, we are interested by the number of particles that have always been below the curve f. For the underlying one-dimensional random walk $(S_k, k \geq 0)$, we then define

$$\tau_f := \inf\{k \geq 0 \,:\, S_k \geq f_k \text{ or } S_k \leq 0\}\,. \tag{5.1}$$

Proposition 5.1. *The following two inequalities hold. For any $x \geq 0$,*

$$P(Z_n > 0) \;\leq\; \mathbb{Q}(\tau_f > n) + \sum_{k=1}^{n} \mathbb{Q}(\tau_f \geq k)\mathrm{e}^{-\nu f_k}\,, \tag{5.2}$$

$$P^x(Z_n > 0) \;\geq\; \frac{\mathbb{Q}(\tau_f > n)\mathrm{e}^{\nu x - 2\nu f_n}}{1 + \sum_{k=0}^{n-1} \sup_{0\leq y\leq f_k}\left\{\mathrm{e}^{\nu y}\mathbb{Q}^y\left(\tau_f^k > n-k\right)\right\}}\,. \tag{5.3}$$

with the notation of (5.1) applied to the sequence $f^k : \ell \mapsto f^k_\ell := f_{k+l}$.

Proof. By the Cauchy-Schwartz inequality,

$$E\left[Z_n(\mathcal{S})\right]^2 \leq E\left[Z_n(\mathcal{S})^2\right] P\left(Z_n(\mathcal{S}) > 0\right)\,,$$

which yields

$$P(Z_n > 0) \geq P(Z_n(\mathcal{S}) > 0) \geq \frac{E[Z_n(\mathcal{S})]^2}{E[Z_n(\mathcal{S})^2]}\,. \tag{5.4}$$

We observe that

$$Z_n(\mathcal{S})^2 \;=\; \sum_{|u|=n} \mathbb{1}_{\{u\in\mathcal{S}\}} Z_n(\mathcal{S}) \tag{5.5}$$

For any $v \in \mathcal{S}$, we define

$$Z_n^v(\mathcal{S}) := \sum_{|u|=n, u>v} \mathbb{1}_{\{u\in\mathcal{S}\}}\,.$$

Moreover, if w is a child of v, and v_i denotes the i-th child of v, we set

$$Z_n^v(\mathcal{S}, w) := \sum_{i, v_i \neq w} Z_n^{v_i}(\mathcal{S})$$

which stands for the number of descendants of v in generation n who have never been beyond the curve f neither below zero and who are not descendant of w. Let u be an individual in generation n and u_ℓ be as previously the ancestor of u at generation ℓ. We have

$$Z_n(\mathcal{S}) \quad = \quad 1 + \sum_{k=0}^{n-1} Z_n^{u_k}(\mathcal{S}, u_{k+1}) \,,$$

from which equation (5.5) becomes

$$\begin{aligned} Z_n(\mathcal{S})^2 &= Z_n(\mathcal{S}) + \sum_{|u|=n} \sum_{k=0}^{n-1} \mathbb{1}_{\{u \in \mathcal{S}\}} Z_n^{u_k}(\mathcal{S}, u_{k+1}) \\ &= Z_n(\mathcal{S}) + \sum_{k=1}^{n} \sum_{|v|=k} Z_n^v(\mathcal{S}) Z_n^{\overleftarrow{v}}(\mathcal{S}, v) \,. \end{aligned}$$

Conditionally on $\overleftarrow{v} \in \mathcal{S}$ and $S(\overleftarrow{v})$, the random variables $Z_n^v(\mathcal{S})$ and $Z_n^{\overleftarrow{v}}(\mathcal{S}, v)$ are independent. This implies that

$$\begin{aligned} E^x[Z_n(\mathcal{S})^2] &= E^x[Z_n(\mathcal{S})] + \sum_{k=1}^{n} \sum_{|v|=k} E^x \left[E[Z_n^v(\mathcal{S}) \,|\, \overleftarrow{v} \in \mathcal{S}, S(\overleftarrow{v})] E[Z_n^{\overleftarrow{v}}(\mathcal{S}, v) \,|\, \overleftarrow{v} \in \mathcal{S}, S(\overleftarrow{v})] \right] \\ &\le E^x[Z_n(\mathcal{S})] + \sum_{k=1}^{n} \sum_{|v|=k} E^x[E[Z_n^{\overleftarrow{v}}(\mathcal{S}) \,|\, \overleftarrow{v} \in \mathcal{S}, S(\overleftarrow{v})]^2] \\ &\le E^x[Z_n(\mathcal{S})] + \sum_{k=0}^{n-1} \sum_{|v|=k} E^x[E[Z_n^v(\mathcal{S}) \,|\, v \in \mathcal{S}, S(v)]^2] \,. \end{aligned} \tag{5.6}$$

For $|v| = k$, we notice that

$$E\left[Z_n^v(\mathcal{S}) \,|\, v \in \mathcal{S},\, S(v)\right] = \mathbb{1}_{\{v \in \mathcal{S}\}} b^{n-k} P^{S(v)}\left(\tau_{f^k} > n-k\right) \,, \tag{5.7}$$

By the usual change of probability, we get for any $a > 0$ and $\ell \ge 0$,

$$\begin{aligned} P^a\left(\tau_{f^k} > \ell\right) &= \mathrm{e}^{\nu a} E[\mathrm{e}^{\nu X_1}]^\ell E_{\mathbb{Q}}^a \left[\mathbb{1}_{\{\tau_{f^k} > \ell\}} \mathrm{e}^{-\nu S_\ell}\right] \\ &\le \mathrm{e}^{\nu a} E[\mathrm{e}^{\nu X_1}]^\ell \mathbb{Q}^a\left(\tau_{f^k} > \ell\right) \\ &= \mathrm{e}^{\nu a} b^{-\ell} \mathbb{Q}^a\left(\tau_{f^k} > \ell\right) \,. \end{aligned}$$

Therefore, equation (5.7) says that

$$E\left[Z_n^v(\mathcal{S}) \,|\, v \in \mathcal{S},\, S(v)\right] \leq \mathbb{1}_{\{v\in\mathcal{S}\}} \mathrm{e}^{\nu S(v)} \mathbb{Q}^{S(v)}\left(\tau_{f^k} > n-k\right) .$$

From (5.6), we deduce that

$$E^x[Z_n^2(\mathcal{S})] \;\leq\; E^x\left[Z_n(\mathcal{S})\right] + \sum_{k=0}^{n-1} \sum_{|v|=k} E^x\left[\mathbb{1}_{\{v\in\mathcal{S}\}}\left(\mathrm{e}^{\nu S(v)} \mathbb{Q}^{S(v)}\left(\tau_{f^k} > n-k\right)\right)^2\right] .$$

We notice that

$$(5.8) \qquad E^x[Z_n(\mathcal{S})] = b^n P^x(\tau_f > n) = E_{\mathbb{Q}^x}\left[\mathrm{e}^{-\nu(S_n - x)} \mathbb{1}_{\{\tau_f > n\}}\right] \leq \mathrm{e}^{\nu x} \mathbb{Q}^x(\tau_f > n) .$$

Consequently,

$$\begin{aligned} E^x[Z_n^2(\mathcal{S})] &\leq \mathbb{Q}^x(\tau_f > n) + \sum_{k=0}^{n-1} b^k E^x\left[\mathbb{1}_{\{\tau_f > k\}}\left(\mathrm{e}^{\nu S_k} \mathbb{Q}^{S_k}\left(\tau_{f^k} > n-k\right)\right)^2\right] \\ &= \mathbb{Q}^x(\tau_f > n) + \sum_{k=0}^{n-1} E_{\mathbb{Q}^x}\left[\mathbb{1}_{\{\tau_f > k\}} \mathrm{e}^{\nu(S_k + x)}\left(\mathbb{Q}^{S_k}\left(\tau_{f^k} > n-k\right)\right)^2\right] . \end{aligned}$$

For any k, we compute that

$$\begin{aligned} & E_{\mathbb{Q}^x}\left[\mathbb{1}_{\{\tau_f > k\}} \mathrm{e}^{\nu S_k}\left(\mathbb{Q}^{S_k}\left(\tau_{f^k} > n-k\right)\right)^2\right] \\ \leq\; & E_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_f > k\}} \mathbb{Q}^{S_k}\left(\tau_{f^k} > n-k\right)\right] \sup_{0\leq y \leq f_k}\left\{\mathrm{e}^{\nu y} \mathbb{Q}^y\left(\tau_{f^k} > n-k\right)\right\} . \end{aligned}$$

Since

$$E_{\mathbb{Q}^x}\left[\mathbb{1}_{\{\tau_f > k\}} \mathbb{Q}^{S_k}\left(\tau_{f^k} > n-k\right)\right] = \mathbb{Q}^x(\tau_f > n) ,$$

it finally gives

$$E_{\mathbb{Q}^x}\left[\mathbb{1}_{\{\tau_f > k\}} \mathrm{e}^{\nu S_k}\left(\mathbb{Q}^{S_k}\left(\tau_{f^k} > n-k\right)\right)^2\right] \leq \mathbb{Q}^x(\tau_f > n) \sup_{0\leq y\leq f_k}\left\{\mathrm{e}^{\nu y} \mathbb{Q}^y\left(\tau_{f^k} > n-k\right)\right\} .$$

Hence,

$$E^x[Z_n(\mathcal{S})^2] \leq \mathbb{Q}^x(\tau_f > n)\left(1 + \sum_{k=0}^{n-1} \sup_{0\leq y\leq f_k}\left\{\mathrm{e}^{\nu y} \mathbb{Q}^y\left(\tau_{f^k} > n-k\right)\right\}\right) .$$

Then, by (5.4) and (5.8),

$$
\begin{aligned}
P^x(Z_n > 0) &\geq \frac{E_{\mathbb{Q}^x}\left[\mathrm{e}^{-\nu(S_n - x)}\mathbb{1}_{\{\tau_f > n\}}\right]^2}{\mathrm{e}^{\nu x}\mathbb{Q}^x(\tau_f > n)\left(1 + \sum_{k=0}^{n-1}\sup_{0\leq y\leq f_k}\left\{\mathrm{e}^{\nu y}\mathbb{Q}^y\left(\tau_{f^k} > n-k\right)\right\}\right)} \\
&\geq \frac{\mathrm{e}^{\nu x - 2\nu f_n}\mathbb{Q}(\tau_f > n)}{1 + \sum_{k=0}^{n-1}\sup_{0\leq y\leq f_k}\left\{\mathrm{e}^{\nu y}\mathbb{Q}^y\left(\tau_{f^k} > n-k\right)\right\}}\,.
\end{aligned}
$$

Turning to the upper bound, we observe that

$$
\{Z_n > 0\} \quad \subset \quad \{Z_n(\mathcal{S}) > 0\}\bigcup_{k=1}^{n} E_k\,.
$$

where E_k is the event that a particle u surviving at time n went beyond the curve f for the first time at time $k < n$. We say then that u is k-good. We already have

$$
P(Z_n(\mathcal{S}) > 0) \leq E[Z_n(\mathcal{S})] \leq \mathbb{Q}(\tau_f > n)\,.
$$

For any $k \leq n$, we observe that

$$
P\left(E_k\right) \leq E\left[\sum_{|u|=k}\mathbb{1}_{\{u \text{ is k-good}\}}\right] = b^k P\left(\tau_0 > n,\, \tau_f = k\right) \leq b^k P\left(\tau_0 > k,\, \tau_f = k\right)\,.
$$

This leads to

$$
P\left(E_k\right) \leq E_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_0 > k\}}\mathbb{1}_{\{\tau_f = k\}}\mathrm{e}^{-\nu S_k}\right] \leq \mathbb{Q}(\tau_f \geq k)\mathrm{e}^{-\nu f_k}\,.
$$

Finally,

$$
P(Z_n > 0) \leq \mathbb{Q}(\tau_f > n) + \sum_{k=1}^{n}\mathbb{Q}(\tau_f \geq k)\mathrm{e}^{-\nu f_k}\,.
$$

$\square$

Let us turn to the proof of Theorem 1.2.

We first obtain the upper bound.

Proof of the upper bound in Theorem 1.2. Let $d := \left(\frac{3\pi^2\sigma^2}{2\nu}\right)^{1/3}$, and define

$$
\begin{aligned}
L(t) &:= d\,(1-t)^{1/3}\,, \forall 0 \leq t \leq 1, \\
f_k &:= d\,(n-k)^{1/3} = n^{1/3}L\left(\frac{k}{n}\right), \forall 1 \leq k \leq n.
\end{aligned}
$$

By Proposition 5.1, it is enough to bound $R(n) := \mathbb{Q}^x(\tau_f > n) + \sum_{k=1}^n \mathbb{Q}^x(\tau_f > k)\mathrm{e}^{-\nu f_k}$. We observe that by (5.2)

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n^{1/3}} \log\left(P^x(Z_n > 0)\right) &\le \limsup_{n\to\infty} R(n) \\ &\le \max(R_1, R_2) =: R_1 \vee R_2\,, \end{aligned}$$

where

$$\begin{aligned} R_1 &:= \limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbb{Q}^x(\tau_f > n)\,, \\ R_2 &:= \max_{0\le k\le n-1} \limsup_{n\to\infty} \frac{1}{n^{1/3}} \log\left(\mathbb{Q}^x(\tau_f \ge k)\mathrm{e}^{-\nu f_k}\right)\,. \end{aligned}$$

By Lemma 4.6 with $\beta(n) = 0$ and $\gamma(n) = n$ (Lemma 4.4 does not suffice because x may be other than 0) and Remark 4.10,

$$R_1 \le -C_{0,L} = -\nu d.$$

Set $\beta^* = 0$, $B(0,n) = 0$. For any $\gamma^* \in (0,1]$ and $n \ge 1$, define $C(\gamma^*, n) := \lfloor \gamma n \rfloor - 1$. Proposition 4.7 and Remark 4.10 yield that, uniformly in $\gamma \in (0,1]$,

$$\lim_{n\to\infty} \sup_{\gamma^*} \left(\frac{1}{n^{1/3}} \log(\mathbb{Q}(\tau_f > C(\gamma^*, n))) + C_{0,L}^{0,\gamma^*} \right) = 0,$$

where

$$C_{0,L}^{0,\gamma^*} = \frac{\pi^2\sigma^2}{2} \int_0^{\gamma^*} \frac{\mathrm{d}t}{L(t)^2} = \left[\frac{3\pi^2\sigma^2(1-t)^{1/3}}{2d^2} \right]_0^{\gamma^*} = \nu d\left(1 - (1-\gamma^*)^{1/3}\right).$$

According to Lemma 4.9, this may be written,

$$\lim_{n\to\infty} \sup_{1\le k\le n} \left(\frac{1}{n^{1/3}} \log(\mathbb{Q}(\tau_f \ge k)) C_{0,L}^{0,\frac{k-1}{n}} \right) = 0.$$

We also have

$$\frac{-\nu f_k}{n^{1/3}} = \nu d \left(1 - \frac{k}{n}\right)^{1/3}.$$

Combining, we obtain

$$R_2 \le -\nu d.$$

Finally,

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} P^x(Z_n > 0) \le -\nu d = -\left(\frac{3\pi^2\nu^2\sigma^2}{2}\right)^{1/3}.$$

□

In the proof of the lower bound, since $x \mapsto P^x(Z_n > 0)$ is nondecreasing, we may assume without loss of generality that $x = 0$. Set $\varepsilon > 0$. As a consequence of the Markov property, we have

$$P^0(\tau_0 > n) \geq P^0(\exists u \in \mathcal{T}, V(u) \geq \varepsilon n^{1/3}\,,\, V(u_i) > 0, \forall 1 \leq i \leq |u|) P^{\varepsilon n^{1/3}}(\tau_0 > n). \tag{5.9}$$

The first factor is controlled by the following lemma :

Lemma 5.2. *Let $\varepsilon > 0$. Let $\eta > 0$ such that $P(X \geq \eta) > 0$. Then*

$$P^0(\exists u \in \mathcal{T}, V(u) \geq \varepsilon n^{1/3}\,,\, V(u_i) > 0, \forall 1 \leq i \leq |u|) \geq P(X \geq \eta)^{\left\lceil \frac{\varepsilon n^{1/3}}{\eta} \right\rceil}.$$

Proof. Pick one individual u in generation $\left\lceil \frac{\varepsilon n^{1/3}}{\eta} \right\rceil$. The right-hand side of the inequality is the probability of the event $\{\forall 1 \leq i \leq |u|, X_u \geq \eta\}$. On this event, it is clear that u survives and $V(u) \geq \varepsilon n^{1/3}$). The lemma follows. □

Actually the lemma above allows us to assume that the original position is high enough (of the order of $n^{1/3}$) so that the condition $L_1(0) < 0$ (which we need for the lower bound of Mogul'skii's estimate) holds.

Proof of the lower bound in Theorem 1.2. From no on we focus on the second factor in the right-hand side of (5.9). We apply Proposition 5.1 with

$$\begin{aligned} L(t) &:= d\,(1-t)^{1/3}\,, \forall 0 \leq t \leq 1, \\ f_k &:= d\,(n-k)^{1/3} + \varepsilon n^{1/3} = n^{1/3}\left(\varepsilon + L\left(\frac{k}{n}\right)\right), \forall 1 \leq k \leq n. \end{aligned}$$

We have $x = \varepsilon n^{1/3} = f_n$. (5.3) yields

$$\begin{aligned} & \frac{1}{n^{1/3}} \log(P^{\varepsilon n^{1/3}}(\tau_0 > n)) \\ \geq \;& \frac{1}{n^{1/3}} \log(\mathbb{Q}^{\varepsilon n^{1/3}}(\tau_f > n)) - \frac{1}{n^{1/3}} \log\left(1 + \sum_{k=0}^{n-1} \sup_{0 \leq x \leq f_k} \{\mathrm{e}^{\nu x} \mathbb{Q}^x\,(\tau_f > n-k)\}\right) - \varepsilon \\ (5.10) = \;& T_1^n - T_2^n - \varepsilon\,, \end{aligned}$$

with obvious notation.
Lemma 4.4, with $L_1^n = L_1 = -\varepsilon$ and $L_2 = L$, yields

$$\lim_{n \to \infty} T_1^n = -C_{-\varepsilon, L}.$$

Similarly, with the same functions and $\forall 0 \le \beta^* < 1$, $B(\beta^*, n) := \lfloor \beta^* n \rfloor$, $\gamma^* = 1$, $C(1,n) = n$, we obtain from Proposition 4.7 and Lemma 4.9 that :

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \left(\max_{0\le k\le n-1} \left(\sup_{0\le x\le f_k} \mathbb{Q}^x \left(\tau_f^k > n-k \right) + C_{-\varepsilon,L}^{\frac{k}{n},1} \right) \right) = 0.$$

Hence

$$\begin{aligned} T_2^n &= o(1) + \max\left\{0, \max_{0\le k\le n-1} \left(\nu L\left(\frac{k}{n}\right) + \nu\varepsilon - C_{-\varepsilon,L}^{\frac{k}{n},1} \right) \right\} \\ &\le o(1) + \max\left\{0, \max_{0\le\alpha\le 1} \left(\nu L(\alpha) + \nu\varepsilon - C_{-\varepsilon,L}^{\alpha,1} \right) \right\}. \end{aligned}$$

As a consequence, $\limsup T_2^n \le \max\{0, T_2\}$ where

$$T_2 := \max_{0\le\alpha\le 1} \left(\nu L(\alpha) + \nu\varepsilon - C_{-\varepsilon,L}^{\alpha,1} \right).$$

A simple computation gives

$$\forall 0 \le \alpha \le 1, \quad C_{0,L}^{\alpha,1} = \nu L(\alpha). \tag{5.11}$$

Therefore, as $\varepsilon \to 0$, $T_2 \to \max_{0\le\alpha\le 1} \left(\nu L(\alpha) - C_{0,L}^{\alpha,1} \right) = 0$. In other words,

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} T_2^n = 0.$$

Taking $\alpha = 0$ in (5.11), we obtain

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} T_1^n = -C_{0,L} = -\nu L(0) = -\nu d.$$

Combining the two last displays with (5.10), we get

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n^{1/3}} \log P^{\varepsilon n^{1/3}}(\tau_0 > n) \ge -\nu d = -\left(\frac{3\pi^2\nu^2\sigma^2}{2} \right)^{1/3}.$$

By Lemma 5.2,

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n^{1/3}} \log P^0(\exists u \in \mathcal{T}, V(u) \ge \varepsilon n^{1/3}, V(u_i) > 0, \forall 1 \le i \le |u|) = 0.$$

Thanks to the two last estimates, we complete the proof of the lower bound by letting $\varepsilon \to 0$ in (5.9). $\square$

6 Appendix

We keep the notation of Section 3.

Lemma 6.1. *For any $\alpha > 1/2$, there exists a constant $d > 0$ such that for any integer $k \geq 1$ and any real $z \geq k$,*

$$\mathbb{Q}^0(S_k \leq -z^\alpha) \leq dk\mathrm{e}^{-z^{\mu(\alpha)}} \tag{6.1}$$

with $\mu(\alpha) := \min(\alpha - 1/2, 1/2)$.

Proof. We closely follow the proof of Lemma 6.3 in [29]. Let $d_2 > 0$ such that $d1 := E^0_{\mathbb{Q}}[\mathrm{e}^{-d_2X}] < \infty$. By our assumptions, it means $d_2 \in (0, \nu - s)$. By the Markov inequality, we have for any $z \geq 0$,

$$\mathbb{Q}^0(X < -z) \leq d_1\mathrm{e}^{-d_2z}\,.$$

We observe that

$$\begin{aligned}
\mathbb{Q}^0\left(S_k \leq -z^\alpha\right) &\leq k\mathbb{Q}^0\left(X \leq -\sqrt{z}\right) + \mathbb{Q}^0\left(S_k \leq -z^\alpha,\, X_i > -\sqrt{z},\, i = 1\ldots k\right) \\
&\leq d_1k\mathrm{e}^{-d_2\sqrt{z}} + \mathrm{e}^{-z^{\alpha-1/2}}E^0_{\mathbb{Q}}\left[\mathrm{e}^{-S_k/\sqrt{z}}\mathbb{1}_{\{X_i > -\sqrt{z},\, 1\leq i\leq k\}}\right] \\
&\leq d_1k\mathrm{e}^{-d_2\sqrt{z}} + \mathrm{e}^{-z^{\alpha-1/2}}E_{\mathbb{Q}}\left[\mathrm{e}^{-X/\sqrt{z}}\mathbb{1}_{\{X > -\sqrt{z}\}}\right]^k.
\end{aligned}$$

The inequality $\mathrm{e}^u \leq 1 + u + u^2$ for $u \leq 1$ implies that

$$\begin{aligned}
E_{\mathbb{Q}}\left[\mathrm{e}^{-X/\sqrt{z}}\mathbb{1}_{\{X > -\sqrt{z}\}}\right] &\leq 1 + E_{\mathbb{Q}}\left[\frac{-X}{\sqrt{z}}\mathbb{1}_{\{X > -\sqrt{z}\}}\right] + E_{\mathbb{Q}}\left[\frac{X^2}{z}\mathbb{1}_{\{X > -\sqrt{z}\}}\right] \\
&= 1 + E_{\mathbb{Q}}\left[\frac{X^2}{z}\mathbb{1}_{\{X > -\sqrt{z}\}}\right] \\
&\leq 1 + \frac{E_{\mathbb{Q}}\left[X^2\right]}{z}\,.
\end{aligned}$$

For $z \geq k$, we get

$$\begin{aligned}
E_{\mathbb{Q}}\left[\mathrm{e}^{-X/\sqrt{z}}\mathbb{1}_{\{X > -\sqrt{z}\}}\right]^k &\leq \left(1 + \frac{E_{\mathbb{Q}}\left[X^2\right]}{z}\right)^k \\
&\leq \left(1 + \frac{E_{\mathbb{Q}}\left[X^2\right]}{z}\right)^z \\
&\leq \exp\left(E_{\mathbb{Q}}\left[X^2\right]\right)\,.
\end{aligned}$$

It yields that

$$\mathbb{Q}^0\left(S_k \leq -z^\alpha\right) \leq d_1k\mathrm{e}^{-d_2\sqrt{z}} + \mathrm{e}^{-z^{\alpha-1/2}}\exp\left(E_{\mathbb{Q}}\left[X^2\right]\right)$$

which completes the proof. □

Lemma 6.2. *For any $\alpha > 1/2$ and $M > 0$, there exists positive constants d_3 and d_4 such that for any $n \geq 1$ and any k between 1 and n, we have*

$$P^x(S_{n-k} \geq M + k^\alpha,\, \tau_0 > n, S_n \leq M) \leq d_3 \gamma^n n^{-3/2} \mathrm{e}^{-k^{d_4}}.$$

Proof. Let $\alpha > 1/2$ and $M > 0$. We define for any $z \geq 0$,

$$\rho_k(z) := P^z(\tau_0 > k,\, S_k \leq M).$$

By the Markov property, we have

$$P^x(S_{n-k} \geq M + k^\alpha,\, \tau_0 > n,\, S_n \leq M) = E^x\left[\rho_k(S_{n-k}), S_{n-k} \geq M + k^\alpha,\, \tau_0 > n-k\right].$$

We observe that

$$\begin{aligned} \rho_k(z) &= \mathrm{e}^{\nu z}\gamma^k E^z_{\mathbb{Q}}\left[\mathrm{e}^{-\nu S_k}, \tau_0 > k,\, S_k \leq M\right] \\ &\leq \mathrm{e}^{\nu z}\gamma^k \mathbb{Q}^z(\tau_0 > k, S_k \leq M) \\ &\leq \mathrm{e}^{\nu z}\gamma^k \mathbb{Q}^z(S_k \leq M). \end{aligned}$$

Since $\mathbb{Q}^z(S_k \leq M) = \mathbb{Q}^0(S_k \leq M - z)$, Lemma 6.1 implies that for any $z \geq M + k^\alpha$,

$$\rho_k(z) \leq \mathrm{e}^{\nu z}\gamma^k d_5 k \exp\left(-(z-M)^{\mu(\alpha)/\alpha}\right).$$

Therefore,

$$\begin{aligned} & P^x(S_{n-k} \geq M + k^\alpha,\, \tau_0 > n,\, S_n \leq M) \\ \leq\; & \gamma^k d_5 k E^x\left[\mathrm{e}^{\nu S_{n-k}} \exp\left(-(S_{n-k}-M)^{\mu(\alpha)/\alpha}\right),\, S_{n-k} \geq M + k^\alpha,\, \tau_0 > n-k\right] \\ =\; & \gamma^n d_5 k \mathrm{e}^{\nu x} E^x_{\mathbb{Q}}\left[\exp\left(-(S_{n-k}-M)^{\mu(\alpha)/\alpha}\right),\, S_{n-k} \geq M + k^\alpha,\, \tau_0 > n-k\right]. \end{aligned}$$

For $k \geq \sqrt{n}$, we get

$$\begin{aligned} P^x(S_{n-k} \geq M + k^\alpha,\, \tau_0 > n,\, S_n \leq M) &\leq \gamma^n d_5 k \exp\left(-(k^\alpha - M)^{\mu(\alpha)/\alpha}\right) \\ &\leq d_6 \gamma^n n^{-3/2} \exp\left(-k^{d_7}\right) \end{aligned}$$

for some $d_6, d_7 > 0$. Suppose then that $k < \sqrt{n}$. We show that we can restrict ourselves to $S_{n-k} \leq M + n^{1/3}$. Indeed,

$$\begin{aligned} & P^x(S_{n-k} \geq M + n^{1/3},\, S_n \leq M,\, \tau_0 > n) \\ =\; & \gamma^n \mathrm{e}^{\nu x} E^x_{\mathbb{Q}}\left[\mathrm{e}^{-\nu S_n}, S_{n-k} \geq M + n^{1/3}, S_n \leq M,\, \tau_0 > n\right] \\ \leq\; & \gamma^n \mathrm{e}^{\nu x} \mathbb{Q}^x(S_{n-k} \geq M + n^{1/3}, S_n \leq M) \\ \leq\; & \gamma^n \mathrm{e}^{\nu x} \mathbb{Q}^0(S_k \leq -n^{1/3}). \end{aligned}$$

We use Lemma 6.1 with $z = \sqrt{n}$ and $\alpha = 2/3$. It yields that $\mathbb{Q}^0(S_k \le -n^{1/3}) \le d_8 k \mathrm{e}^{-n^{1/6}}$. We deduce that

$$P^x(S_{n-k} \ge M + n^{1/3},\, S_n \le M,\, \tau_0 > n) \le d_9 \gamma^n \mathrm{e}^{\nu x} n^{-3/2} \mathrm{e}^{-k^{d_{10}}}$$

for some $d_9, d_{10} > 0$. It remains to bound the probability that $S_{n-k} \in [M + k^\alpha, M + n^{1/3}]$. We have

$$\begin{aligned} & P^x(S_{n-k} \in [M + k^\alpha, n^{1/3}],\, \tau_0 > n,\, S_n \le M) \\ \le\ & \gamma^n d_5 k \mathrm{e}^{\nu x} E^x_{\mathbb{Q}} \left[\exp\left(-(S_{n-k} - M)^{\mu(\alpha)/\alpha} \right),\, S_{n-k} \ge M + k^\alpha,\, \tau_0 > n - k \right]. \end{aligned}$$

Reasoning on the value of S_{n-k}, we get

$$\begin{aligned} & E^x_{\mathbb{Q}} \left[\exp\left(-(S_{n-k} - M)^{\mu(\alpha)/\alpha} \right),\, S_{n-k} \ge M + k^\alpha,\, \tau_0 > n - k \right] \\ \le\ & \sum_{\ell = \lfloor M + k^\alpha \rfloor}^{\lfloor M + n^{1/3} \rfloor} \exp\left(-(\ell - M)^{\mu(\alpha)/\alpha} \right) \mathbb{Q}^x(S_{n-k} \in [\ell, \ell+1), \tau_0 > n - k). \end{aligned}$$

By Lemma 2.3, there exists a constant cd_{11} such that for any $m \ge 1$ and any $\ell \le M + n^{1/3}$, we have

$$\mathbb{Q}^x(S_m \in [\ell, \ell+1), \tau_0 > m) \le d_{11} m^{-3/2} (1 + \ell)^2,$$

where we used the fact that the renewal function behaves linearly at infinity. We deduce that

$$\begin{aligned} & E^x_{\mathbb{Q}} \left[\exp\left(-(S_{n-k} - M)^{\mu(\alpha)/\alpha} \right),\, S_{n-k} \ge M + k^\alpha,\, \tau_0 > n - k \right] \\ \le\ & d_{11} (n-k)^{-3/2} \sum_{\ell = \lfloor M + k^\alpha \rfloor}^{\lfloor M + n^{1/3} \rfloor} (1 + \ell)^2 \exp\left(-(\ell - M)^{\mu(\alpha)/\alpha} \right) \\ \le\ & d_{12} n^{-3/2} \mathrm{e}^{-k^{d_{13}}} \end{aligned}$$

for some $d_{12}, d_{13} > 0$ since $k < \sqrt{n}$. It completes the proof. □

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Chapter 3

The critical barrier for the survival of the branching random walk with absorption

The critical barrier for the survival of the branching random walk with absorption[1]

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Summary. We study a branching random walk on $\mathbb{R}$ with an absorbing barrier. The position of the barrier depends on the generation. In each generation, only the individuals born below the barrier survive and reproduce. Given a reproduction law, Biggins et al. [15] determined whether a linear barrier allows the process to survive. In this paper, we refine their result : in the boundary case in which the speed of the barrier matches the speed of the minimal position of a particle in a given generation, we add a second order term $an^{1/3}$ to the position of the barrier for the n^{th} generation and find an explicit critical value a_c such that the process dies when $a < a_c$ and survives when $a > a_c$. We also obtain the rate of extinction when $a < a_c$ and a lower bound for the population when it survives.

Key words. Branching random walk, absorption.

AMS subject classifications. 60J80.

1 Introduction

We study a discrete-time branching random walk on $\mathbb{R}$. The population forms a well-known Galton-Watson tree $\mathcal{T}$, and some extra information is added : to each individual $u \in \mathcal{T}$ we attach a displacement $\xi_u \in \mathbb{R}$ from the position of her parent. We set the initial ancestor ϱ at the origin, hence the individual u has position

$$V(u) = \sum_{\varrho < v \leq u} \xi_v = \sum_{i=1}^{|u|} \xi_{u_i},$$

[1]This chapter comes from an article submitted to *Annales de l'Institut Henri Pointcaré*. We improve here results for the critical case that constitute the second half of Chapter 2, and this leads to repeating certain arguments. In particular, Section 2.2 borrows heavily from Section 4 of Chapter 2.

where $|u|$ is the generation of u and u_i the ancestor of u in generation i. We denote by $\mathcal{T}_n := \{u \in \mathcal{T} : |u| = n\}$ the population at time n. We define an infinite path u through $\mathcal{T}$ as a sequence of individuals $u = (u_i)_{i \in \mathbb{N}}$ such that

$$\forall i \in \mathbb{N}, |u_i| = i \text{ and } u_i < u_{i+1}.$$

We denote their collection by $\mathcal{T}_\infty$.

Now we explain how the displacements $\xi_u, u \in \mathcal{T}$ are distributed. A simple choice, with very nice properties would be to take them i.i.d. but actually everything still works in a more general setting. All individuals still reproduce independently and the same way, but we allow correlations in the number and displacements of the children of every single individual. If we write $\Gamma(u)$ for the set of children of u, our requirement is that the point processes $\{\xi_v,\ u \in \Gamma(u)\}$, (with u running over all the potential individuals of the random tree $\mathcal{T}$) are i.i.d.

We define a barrier as a function $\varphi : \mathbb{N} \to \mathbb{R}$. In the branching random walk with absorption, the individuals u such that $V(u) > \varphi(|u|)$, i.e. born above the barrier are removed : they are immediately killed and do not reproduce.

Kesten [61], Derrida and Simon [31],[32], Harris and Harris [48] have studied the continuous analog of this process, the branching Brownian motion with absorption. The understanding of what happens in the continuous setting, more convenient to handle from technical point of view, greatly helps us in the discrete one. In particular, we borrow here some ideas from Kesten [61].

Biggins et al. [15] introduced the branching random walk with an absorbing barrier in order to answer questions about parallel simulations. Pemantle [97] and Gantert et al. [44] also studied this model.

A natural question that arises is whether the process survives. This obviously depends on the walk as well as on the barrier. The case of the linear barriers has been solved by Biggins et al. [15].

Before stating their result, we need to introduce some notation :

We denote the intensity measure of the point process by μ, and its Laplace-Stieljes transform by Φ :

$$\Phi(t) = \mathbb{E}\left[\sum_{|u|=1} \mathrm{e}^{-t\xi_u}\right] = \int_{\mathbb{R}} \mathrm{e}^{-tz}\mu(\mathrm{d}z).$$

We assume that the expected number of children $\Phi(0)$ is finite and that negative displacements occur, i.e. that $\mu((-\infty, 0)) > 0$.

We also define $\Psi = \log \Phi$, this is a strictly convex function that takes values in $(-\infty, +\infty]$. We call critical the case where

$$\Phi(1) = \mathbb{E}\left[\sum_{|u|=1} \mathrm{e}^{-\xi_u}\right] = 1 \text{ and } \Phi'(1) := \mathbb{E}\left[\sum_{|u|=1} \xi_u \mathrm{e}^{-\xi_u}\right] = 0.$$

This can also be written $\Psi(1) = 0$ and $\Psi'(1) = 0$.

Theorem 1.1 (Biggins et al. [15]). *In the critical case, we have :*

$$\mathbb{P}\left(\exists u \in \mathcal{T}_\infty, \forall i \geq 1, V(u_i) \leq i\varepsilon\right) \begin{cases} = 0 \text{ if } \varepsilon \leq 0, \\ > 0 \text{ if } \varepsilon > 0. \end{cases}$$

The aim of this article is to refine this result by replacing the linear barrier $i \mapsto i\varepsilon$ with a more general barrier $i \mapsto \varphi(i)$.

Given a barrier φ we do not know in general whether $\mathbb{P}\left(\exists u \in \mathcal{T}_\infty, \forall i \geq 1, V(u_i) \leq \varphi(i)\right) = 0$ or not. Theorem 1.1 leads us to focus on barriers such that $\frac{\varphi(i)}{i} \to 0$.

Some specific technical difficulties arise in the computation of the second moment (that we use in order to give a lower bound for the survival probability or to prove survival) when dealing with Galton-Watson trees of unbounded degree. Actually individuals with many children may cause trouble. In order to have a sufficient control, we assume from now on that the following condition holds :

$$\exists \delta_1 > 0, \Phi(1+\delta_1) < +\infty \text{ and } \exists \delta_2 > 0, \mathbb{E}\left[\#\mathcal{T}_1^{1+\delta_2}\right] < +\infty. \tag{1.1}$$

In the critical case, provided the number of children is deterministic or condition (1.1) holds, we obtain the following theorem :

Theorem 1.2. *We assume*

$$\sigma^2 := \Phi''(1) = \mathbb{E}\left[\sum_{|u|=1} \xi_u^2 \mathrm{e}^{\xi_u}\right] < +\infty.$$

Let $a_c = \frac{3}{2}\left(3\pi^2\sigma^2\right)^{1/3}$. *Then we have :*

$$\mathbb{P}\left(\exists u \in \mathcal{T}_\infty, \forall i \geq 1, V(u_i) \leq a i^{1/3}\right) \begin{cases} = 0 \text{ if } a < a_c, \\ > 0 \text{ if } a > a_c. \end{cases}$$

Unfortunately, we are not able to conclude in the case $a = a_c$, nor to give a necessary and sufficient condition on a general barrier for a line of descent to survive below it.

Theorem 1.2 has the following corollary :

Corollary 1.3. *Under the hypothesis of Theorem 1.2, we have, almost surely, on the set of ultimate survival of the underlying Galton-Watson process,*

$$\inf_{u \in \mathcal{T}_\infty} \limsup_{n \to \infty} \frac{V(u_n)}{n^{1/3}} = a_c.$$

While proving Theorem 1.2, we actually obtain stronger results. The two following propositions together imply the theorem.

Proposition 1.4. *If $a > a_c$, then the equation $a = b + \frac{3\pi^2\sigma^2}{2b^2}$ has two solutions in b, let b_a be the one such that $b_a > \frac{2a_c}{3}$. For any $\varepsilon > 0$, for any $N \in \mathbb{N}$ large enough, we have with positive probability :*

$$\forall k \geq 1, \#\{u \in \mathcal{T}_{N^k} : \forall i \leq N^k, (a - b_a)i^{1/3} \leq V(u_i) \leq ai^{1/3}\} \geq \exp\left(N^{k/3}(b_a - \varepsilon)\right).$$

Proposition 1.5. *If $a < a_c$, then there exists some constant $c > 0$ such that*

$$\frac{1}{n^{1/3}} \log \mathbb{P}\left(\exists u \in \mathcal{T}_n, \forall i \leq n, V(u_i) \leq ai^{1/3}\right) \to -c.$$

The constant c, which depends on a, is determined in Section 5.

When $0 < a < a_c$, extinction means that the total progeny Z is almost surely finite. This random variable has infinite mean, since the expected number of surviving individuals in generation n is $\exp\left(an^{1/3}(1 + o(1))\right)$. We can estimate the tail of the distribution of Z :

Proposition 1.6. *If $a < a_c$, then let g be the optimal function determined in Section 5, $c := g(0)$ and $d := \max_{[0,1]} g$.*

$$\mathbb{P}(Z > k) = k^{-\frac{c}{d}(1+o(1))}.$$

Remark 1.7. In the case $a \leq 0$, g is decreasing, hence $d = c$ and the claim of Proposition 1.6 is weaker than a known result, conjectured by Aldous and proved by Addario-Berry and Broutin [1], and improved by Aidékon [3] that for $a = 0$, $\mathbb{E}[Z] < +\infty$ and $\mathbb{E}[Z \log Z] = +\infty$. Exponents less than -1 are obtained (in a work in progress by Aidékon, Hu and Zindy) for linear barriers $i \mapsto -\varepsilon i$, which corresponds to what is often referred as the subcritical case.

1.1 About general barriers

Consider a general barrier $\varphi : \mathbb{N} \to \mathbb{R}$. We define

$$a^+ := \limsup_{n \to \infty} \frac{\varphi(n)}{n^{1/3}} \text{ and } a^- := \liminf_{n \to \infty} \frac{\varphi(n)}{n^{1/3}}.$$

We deduce from Theorem 1.2 that there is extinction when $a^+ < a_c$ and survival when $a^- > a_c$. Making some modifications to the computations of Section 3, we can prove the following result :

Theorem 1.8. *Assume $a^+ \geq a_c$. The equation $a^+ = b + \frac{3\pi^2\sigma^2}{2b^2}$ admits a unique solution $b_{a_c} = \frac{2a_c}{3}$ if $a^+ = a_c$, and two solutions if $a > a_c$. Let $b_{a^+} \geq \frac{2a_c}{3}$ be the larger solution.*

If $a^- < \frac{3\pi^2\sigma^2}{2b_{a^+}^2}$, then there is extinction.

We notice that $\frac{3\pi^2\sigma^2}{2b_{a^+}^2} \leq \frac{3\pi^2\sigma^2}{2b_{a_c}^2} = \frac{a_c}{3} < a_c$.

When $a^+ \geq a_c$, $a^- > \frac{3\pi^2\sigma^2}{2b_{a^+}^2}$, there is not always survival. For example, if $\varphi(n)$ equals $a^+n^{1/3}$ for n even and $a^-n^{1/3}$ for n odd, then a^+ does not matter, it is easy to see that there is extinction if $a^- < a_c$: staying below this barrier is almost as difficult as for the barrier $n \mapsto a^-n^{1/3}$. The trouble comes from the fact that $\frac{\varphi(n)}{n^{1/3}}$ is too often close to a^-.

Actually the condition in Theorem 1.8 is sharp in the sense that, if we choose some $a^+ \geq a_c$ and $a^- > \frac{3\pi^2\sigma^2}{2b_{a^+}^2}$, we can construct a barrier φ satisfying $\limsup_{n\to\infty} \frac{\varphi(n)}{n^{1/3}} = a^+$ and $\liminf_{n\to\infty} \frac{\varphi(n)}{n^{1/3}} = a^-$ such that the process survives with positive probability. It suffices to take $\varphi(n) = a^-n^{1/3}$ if $n \in \{N^k : k \in \mathbb{N}\}$ and $\varphi(n) = a^+n^{1/3}$ otherwise, for some integer N big enough, depending on a^+ and a^-. The proof of this is essentially identical to the proof of the lower bound contained in Section 4.

1.2 The reduction to the critical case

It is possible to apply these results to certain non critical branching random walks. We analyze in which cases this reduction is possible in Appendix 7. Let $t > 0$ such that $\Phi(t) < +\infty$ and $\Phi'(t) < +\infty$. We define a new branching random walk by changing the position of the individual z into $\widetilde{V}(z) = tV(z) + \Psi(t)|z|$ for all $z \in \mathcal{T}$.

Then, with obvious notation, a straightforward computation gives :

$$\widetilde{\Phi}(1) = \mathbb{E}\left[\sum_{|u|=1} \mathrm{e}^{-\widetilde{\xi}_u}\right] = 1;$$

$$\widetilde{\Phi}'(1) = -\mathbb{E}\left[\sum_{|u|=1} \widetilde{\xi}_u \mathrm{e}^{-\widetilde{\xi}_u}\right] = t\frac{\Phi'(t)}{\Phi(t)} - \Psi(t) = t\Psi'(t) - \Psi(t).$$

When we can find $t^* > 0$ such that :

$$t^*\Psi'(t^*) - \Psi(t^*) = 0, \tag{1.2}$$

then the new branching random walk is critical and we can apply Theorem 1.2, provided $\widetilde{\sigma}^2 := \mathbb{E}\left[\sum_{|u|=1} \widetilde{\xi}_u^2 \mathrm{e}^{-\widetilde{\xi}_u}\right] = t^{*2}\Phi''(t^*) - \Psi(t^*)^2$ is finite. This last condition, which is equivalent to $\Phi''(t^*) < \infty$ will always be fulfilled when $\exists t > t^*, \Phi(t) < +\infty$.
The strict convexity of Ψ implies that, when it exists, t^* is unique.
The existence of t^* is discussed in Section 7.

The rest of the paper is organized as follows :
Section 2 introduces the tools we will use in the proof of our main results.
Section 3 is devoted to the proof of the upper bound in Proposition 1.5, which contains the first part of Theorem 1.2.
In Section 4, we prove Proposition 1.4 which implies the second part of Theorem 1.2.
In Section 5, we complete the proof of Proposition 1.5. We skip many details of technical arguments already exposed in Section 4 to obtain the lower bound and go back over some results of Section 3 in order to prove that the two bounds agree.
In Section 6, we prove Theorem 1.8, Proposition 1.6 and Corollary 1.3.

2 Some preliminaries

2.1 Many-to-one lemma

Since $\mathbb{E}\left[\sum_{|u|=1} \mathrm{e}^{-\xi_u}\right] = 1$, we can define the law of a random variable X such that for any measurable nonnegative function f,

$$\mathbb{E}[f(X)] = \mathbb{E}\left[\sum_{|u|=1} \mathrm{e}^{-\xi_u} f(\xi_u)\right].$$

Then $\mathbb{E}[X] = \mathbb{E}\left[\sum_{|u|=1} \xi_u \mathrm{e}^{-\xi_u}\right]$ so that X is centered by hypothesis.
We denote

$$\sigma^2 := \mathbb{E}[X^2] = \Phi''(1).$$

Let $(X_i)_{i\in\mathbb{N}^*}$ be a i.i.d. sequence of copies of X. Write for any $n \in \mathbb{N}$, $S_n := \sum_{0<i\leq n} X_i$. S is then a mean-zero random-walk starting from the origin.

We can now state the many-to-one lemma : (this is exactly Lemma 4.1 (iii) of Biggins-Kyprianou [12]) :

Lemma 2.1 (Biggins-Kyprianou [12]). *For any $n \geq 1$ and any measurable function $F : \mathbb{R}^n \to [0, +\infty)$,*

$$\mathbb{E}\left[\sum_{|u|=n} \mathrm{e}^{-V(u)} F(V(u_i), 1 \leq i \leq n)\right] = \mathbb{E}\left[F(S_i, 1 \leq i \leq n)\right].$$

The proof of the lower bound for the survival probability also requires the following bivariate version of the many-to-one lemma.

Lemma 2.2 (Gantert-Hu-Shi [44]). *Let (X, ν) be a random variable taking values in $\mathbb{R} \times \mathbb{N}^*$ such that for any measurable nonnegative function f,*

$$\mathbb{E}[f(X, \nu)] = \mathbb{E}\left[\sum_{|u|=1} \mathrm{e}^{-\xi_u} f(\xi_u, \mathcal{T}_1)\right].$$

Let $n \geq 1$ and $(X_i, \nu_i)_{1 \leq i \leq n}$ be i.i.d. copies of (X, ν). Write for any $0 \leq k \leq n$, $S_k := \sum_{0 < i \leq k} X_i$. Then for any measurable function $F : (\mathbb{R} \times \mathbb{N}^*)^n \to [0, +\infty)$,

$$\mathbb{E}\left[\sum_{|u|=n} \mathrm{e}^{-V(u)} F(V(u_i), \#\Gamma(u_{i-1}), 1 \leq i \leq n)\right] = \mathbb{E}\left[F(S_i, \nu_i, 1 \leq i \leq n)\right].$$

The proof, very similar to the one of Lemma 2.1, is omitted.

2.2 Mogul'skii's estimate

For any L, $\widetilde{L} \in \mathcal{F}[0,1]$, we write $L < \widetilde{L}$ when $\forall t \in [0,1], L(t) < \widetilde{L}(t)$ and $L \leq \widetilde{L}$ when $\forall t \in [0,1], L(t) \leq \widetilde{L}(t)$. If $n \geq 1$, we write $L <_n \widetilde{L}$ when $\forall 1 \leq k \leq n$, $L\left(\frac{k}{n}\right) < \widetilde{L}\left(\frac{k}{n}\right)$ and $L \leq_n \widetilde{L}$ when $\forall 1 \leq k \leq n$, $L\left(\frac{k}{n}\right) \leq \widetilde{L}\left(\frac{k}{n}\right)$.

Theorem 2.3 (Mogul'skii). *Let ξ_1, ξ_2, $\ldots$ be i.i.d. random variables such that $E[\xi_1] = 0$ and $\sigma^2 := E[\xi_1^2] < \infty$. Let $(x_n,\, n \geq 0)$ be a sequence of positive numbers such that*

$$\begin{aligned} \lim_{n\to\infty} x_n &= +\infty\,, \\ \lim_{n\to\infty} \frac{x_n}{\sqrt{n}} &= 0\,. \end{aligned}$$

Define for any $n \geq 0$

$$S_n := S_0 + \xi_1 + \xi_2 + \ldots + \xi_n,$$

where $S_0 = z$ almost surely under the probability $\mathbb{P}^z$ ($z \in \mathbb{R}$).

When $z = 0$, write $\mathbb{P} := \mathbb{P}^0$ and define, for any $t \in [0,1]$,

$$s_n(t) := \frac{S_{\lfloor tn \rfloor}}{x_n} = \frac{\xi_1 + \xi_2 + \ldots + \xi_k}{x_n} \qquad \text{for} \qquad k/n \le t < (k+1)/n\,.$$

Then, for any L_1, $L_2 \in \mathcal{C}[0,1]$, with

$$L_1 < L_2 \text{ and } L_1(0) < 0 < L_2(0), \tag{2.1}$$

we have, as $n \to \infty$,

$$\log\left(\mathbb{P}(L_1 < s_n < L_2)\right) \sim -C_{L_1,L_2} n x_n^{-2},$$

where

$$C_{L_1,L_2} := \frac{\pi^2 \sigma^2}{2} \int_0^1 \frac{\mathrm{d}t}{\left(L_2(t) - L_1(t)\right)^2}\,.$$

We keep the notations and assumptions of Theorem 2.3 throughout this section. For the proof, we refer to [91]. The choice of $x_n = n^{1/3}$ is the same throughout the paper, we introduce it as late as possible (see equation (2.2)) because many argument of this section do not depend on it and we prefer the notation x_n when $n^{1/3}$ is the spatial scaling.

Lemma 2.4. *Set L_1, $L_2 \in \mathcal{C}[0,1]$, with $L_1 < L_2$ and $L_1(0) < 0 < L_2(0)$. For any sequences $(L_1^n)_n$ and $(L_2^n)_n$ of $\mathcal{F}[0,1]$ such that $\|L_1^n - L_1\|_\infty \to 0$ and $\|L_2^n - L_2\|_\infty \to 0$ as $n \to \infty$, we have*

$$\log\left(\mathbb{P}(L_1^n < s_n < L_2^n)\right) \sim -n x_n^{-2} C_{L_1,L_2}.$$

Remark 2.5. In the conclusion of Lemma 2.4, we can replace the strict inequalities

$$L_1^n < s_n < L_2^n$$

by weak ones (or take one strict and the other weak) and obtain the same estimate by exactly the same argument. One can easily check that this also applies to the other results of this section since we deduce them from Lemma 2.4.

Proof of Lemma 2.4. Let $0 < \varepsilon < \frac{1}{2}\min_{[0,1]}(L_2 - L_1)$. We can choose $N \ge 1$ such that

$$\forall j \in \{1,2\}, \forall n \ge N, \|L_j^n - L_j\|_\infty < \varepsilon.$$

Then, for any $n \ge N$, we have

$$\{L_1 + \varepsilon < s_n < L_2 - \varepsilon\} \subset \{L_1^n < s_n < L_2^n\} \subset \{L_1 - \varepsilon < s_n < L_2 + \varepsilon\}.$$

Using the corresponding inequalities for probabilities and applying Theorem 2.3, we get

$$-C_{L_1+\varepsilon,L_2-\varepsilon} \leq \liminf_{n\to\infty} \frac{x_n^2}{n} \log \mathbb{P}(L_1^n < s_n < L_2^n) \leq \limsup_{n\to\infty} \frac{x_n^2}{n} \log \mathbb{P}(L_1^n < s_n < L_2^n) \leq -C_{L_1-\varepsilon,L_2+\varepsilon}.$$

We make $\varepsilon \to 0$. Then $C_{L_1+\varepsilon,L_2-\varepsilon} \to C_{L_1,L_2}$ and $C_{L_1-\varepsilon,L_2+\varepsilon} \to C_{L_1,L_2}$, and the lemma is proved. □

Lemma 2.6. *Set L_1, $L_2 \in \mathcal{C}[0,1]$, with $L_1 < L_2$ and $L_1(0) < 0 < L_2(0)$. For any sequences $(L_1^n)_n$ and $(L_2^n)_n$ of $\mathcal{F}[0,1]$ such that $\|L_1^n - L_1\|_\infty \to 0$ and $\|L_2^n - L_2\|_\infty \to 0$ as $n \to \infty$, we have*

$$\log\left(\mathbb{P}(L_1^n <_n s_n <_n L_2^n)\right) \sim -n x_n^{-2} C_{L_1,L_2}.$$

Proof. We introduce the sequences of piecewise constant interpolation functions (a similar argument with the linear interpolation functions also works) $(\widetilde{L}_1^n)_n$ and $(\widetilde{L}_2^n)_n$ defined by :

$$\forall j \in \{1,2\}, \forall n \geq 1, \forall 0 \leq t \leq 1,\ \widetilde{L}_j^n(t) := L_j^n\left(\frac{\lfloor tn \rfloor}{n}\right).$$

We notice that

$$\forall n \geq 1, \{L_1^n <_n s_n <_n L_2^n\} = \{\widetilde{L}_1^n < s_n < \widetilde{L}_2^n\}.$$

Before concluding by applying Lemma 2.4 with $\widetilde{L}_j^n$ playing the role of L_j^n, we have to check that these sequences converge uniformly to L_j as $n \to \infty$ for $j = 1,2$. Set $\varepsilon > 0$. We can choose $N \geq 1$ such that for $j = 1,2$ for any $n \geq N$, we have $\|L_j^n - L_j\|_\infty < \varepsilon$.

We can also choose $\eta > 0$ such that for $j = 1,2$,

$$\forall t, t' \in [0,1], (|t - t'| \leq \eta \Rightarrow |L_j(t) - L_j(t')| < \varepsilon).$$

Let $t \in [0,1]$ and $n \geq \frac{1}{\eta}$. Noticing that $|\frac{\lfloor tn \rfloor}{n} - t| \leq \frac{1}{n}$, we get for $j = 1,2$

$$\forall n \geq \max(N, \frac{1}{\eta}), \left|\widetilde{L}_j^n(t) - L_j(t)\right| \leq \left|L_j^n\left(\frac{\lfloor tn \rfloor}{n}\right) - L_j\left(\frac{\lfloor tn \rfloor}{n}\right)\right| + \left|L_j\left(\frac{\lfloor tn \rfloor}{n}\right) - L_j(t)\right| \leq 2\varepsilon.$$

Then $\|\widetilde{L}_j^n - L_j\|_\infty \to 0$ as $n \to \infty$ and we can apply Lemma 2.4, which proves Lemma 2.6. □

From now on, we set :

$$\forall n \geq 1, x_n := n^{1/3}. \tag{2.2}$$

Lemma 2.7. *Set L_1, $L_2 \in \mathcal{C}[0,1]$, with $L_1 < L_2$ and $L_1(0) < 0 < L_2(0)$. Let $(L_1^n)_n$ and $(L_2^n)_n$ be sequences of $\mathcal{F}[0,1]$ such that $\|L_1^n - L_1\|_\infty \to 0$ and $\|L_2^n - L_2\|_\infty \to 0$ as $n \to \infty$. Let β and γ be positive real numbers such that $0 \le \beta < \gamma \le 1$. Let $u^*, v^* \in \mathbb{R}$ such that $L_1(\beta) \le u^* < v^* \le L_2(\beta)$. Let $(u_n)_n$ and $(v_n)_n$ be sequences of reals satisfying*

$$\frac{u_n}{x_n} \to u^*, \quad \frac{v_n}{x_n} \to v^*, \quad L_1^n(B(n))x_n \le u_n \le v_n \le L_2^n(B(n))x_n \quad \forall n \ge 1.$$

Let $(B(n))_n$ and $(C(n))_n$ be sequences of reals satisfying

$$\frac{B(n)}{n} \to \beta, \quad \frac{C(n)}{n} \to \gamma, \quad 1 \le B(n) < C(n) \le n, \forall n \ge 1.$$

We also assume that :

$$\exists M \in \mathbb{N}^*, \forall m \in \mathbb{N}^*, \#\{n : (C-B)(n) = m\} \le M.$$

It is easy to see that the last condition holds if the sequence $(C(n) - n\gamma - B(n) + n\beta)_n$ is bounded. Then

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log\left(\sup_{u_n\le z\le v_n} \mathbb{P}^z\left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-B(n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(n) < k \le C(n)\right)\right) \le -C^{\beta,\gamma}_{L_1-v^*,L_2-u^*},$$

where, for any functions continuous L_1 and $L_2 : [0,1] \mapsto \mathbb{R}$ such that $L_1 \le L_2$,

$$C^{\beta,\gamma}_{L_1,L_2} := \frac{\pi^2\sigma^2}{2}\int_\beta^\gamma \frac{\mathrm{d}t}{\left(L_2(t) - L_1(t)\right)^2}.$$

Proof. Write $m := C(n) - B(n)$. Notice that

$$\begin{aligned} &\left\{\forall z \in [u_n, v_n], \forall k \le m, x_n L_1^n\left(\frac{B(n)+k}{n}\right) < z + S_k < x_n L_2^n\left(\frac{B(n)+k}{n}\right)\right\} \\ \subset\ &\left\{\forall k \le m, x_n L_1^n\left(\frac{B(n)+k}{n}\right) - v_n < S_k < x_n L_2^n\left(\frac{B(n)+k}{n}\right) - u_n\right\}. \end{aligned} \tag{2.3}$$

Let $A = \{m \in \mathbb{N}^* : \exists n \in \mathbb{N}^*, C(n) - B(n) = m\}$. By hypothesis, each $m \in A$ is attained by $C - B$ for at most M different values of n, hence we can define a surjection $\varphi : \{1, 2, \dots, M\} \times A \mapsto \mathbb{N}^*$ such that for any $1 \le l \le M$, $\varphi(l, m) =: n$ satisfies $C(n) - B(n) = m$.

For each $1 \le l \le M$, we define

$$\widetilde{L}_1^m(t) = \frac{L_1^n\left(\frac{(1-t)B(n)+tC(n)}{n}\right)x_n - v_n}{x_m}, \quad \widetilde{L}_2^m(t) = \frac{L_2^n\left(\frac{(1-t)B(n)+tC(n)}{n}\right)x_n - u_n}{x_m},$$

where $n := \varphi(l,m)$. Using (2.2) and $n \sim (\gamma-\beta)^{-1}m$, we obtain $x_n \sim x_m(\gamma-\beta)^{-1/3}$. Consequently these sequences of functions satisfy, as $m \to \infty$ (and so $n := \varphi(l,m) \to \infty$) :

$$\begin{aligned} \widetilde{L}_1^m(t) \to \widetilde{L}_1(t) &:= (\gamma-\beta)^{-1/3}\left[L_1\left((1-t)\beta + t\gamma\right) - v^*\right], \\ \widetilde{L}_2^m(t) \to \widetilde{L}_2(t) &:= (\gamma-\beta)^{-1/3}\left[L_2\left((1-t)\beta + t\gamma\right) - u^*\right]. \end{aligned}$$

For each $1 \le l \le M$, we apply Lemma 2.6 with $\widetilde{L}_1^m$ and $\widetilde{L}_2^m$ to the probability of the event in the right-hand side of (2.3), we obtain, for the event on the left-hand side, as $m \to \infty$

$$\begin{aligned} &\mathbb{P}\left(\forall z \in [u_n, v_n], \forall k \le m, x_n L_1^n\left(\frac{B(n)+k}{n}\right) < z + S_k < x_n L_2^n\left(\frac{B(n)+k}{n}\right)\right) \\ \le\ & \exp\left(-(1+o(1))m^{1/3}\frac{\pi^2\sigma^2}{2}\int_0^1 \frac{\mathrm{d}t}{\left(\widetilde{L}_2(t)-\widetilde{L}_1(t)\right)^2}\right) \\ =\ & \exp\left(-(1+o(1))n^{1/3}\frac{\pi^2\sigma^2}{2}\int_\beta^\gamma \frac{\mathrm{d}t}{\left(L_2(t)-u^*-L_1(t)+v^*\right)^2}\right) \\ =\ & \exp\left(-(1+o(1))n^{1/3}C^{\beta,\gamma}_{L_1-v^*,L_2-u^*}\right). \end{aligned}$$

This bound holds with n running along the M subsequences $\varphi(l,m)_m$, $1 \le l \le M$, which together cover all the values $n \in \mathbb{N}^*$, and thus Lemma 2.7 is proved. □

Lemma 2.8. *Set L_1, $L_2 \in \mathcal{C}[0,1]$, with $L_1 < L_2$ and $L_1(0) < 0 < L_2(0)$. Let $(L_1^n)_n$ and $(L_2^n)_n$ be sequences of $\mathcal{F}[0,1]$ such that $\|L_1^n - L_1\|_\infty \to 0$ and $\|L_2^n - L_2\|_\infty \to 0$ as $n \to \infty$. Let β and γ be positive real numbers such that $0 \le \beta < \gamma \le 1$. Let $(\beta(n))_n$ and $(C(n))_n$ be sequences of reals satisfying*

$$\frac{B(n)}{n} \to \beta, \quad \frac{C(n)}{n} \to \gamma, \quad 1 \le B(n) < C(n) \le n\, \forall n \ge 1.$$

We also assume :

$$\exists M \in \mathbb{N}^*, \forall m \in \mathbb{N}^*, \#\{n : (C-B)(n) = m\} \le M.$$

It is easy to see that the last condition holds if the sequence $(C(n) - n\gamma - B(n) + n\beta)_n$ is bounded. Then

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log\left(\sup_z \mathbb{P}^z\left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-B(n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(n) < k \le C(n)\right)\right) \le -C^{\beta,\gamma}_{L_1,L_2},$$

where the $\sup_z$ is over the $z \in \mathbb{R}$ such that $x_n L_1^n(B(n)) \le z \le x_n L_2^n(B(n))$.

Proof. Let $\varepsilon > 0$. Let N be an integer such that $N\varepsilon > L_2(\beta) - L_1(\beta)$. We define for $j = 0, 1, \ldots, N$,

$$u_n^j := x_n \frac{L_1^n(\beta)(N-j) + L_2^n(\beta)j}{N}.$$

With obvious notation, we have

$$\sup_{x_n L_1^n(B(n)) \le z \le x_n L_2^n(B(n))} p(z,n) = \max_{0 \le j \le N-1} \sup_{u_n^j \le z \le v_n^j} p(z,n).$$

We apply Lemma 2.7 N times, with $u_n = u_n^j$ and $v_n = u_n^{j+1}$, $J = 0, 1, \ldots, N-1$ and get by the preceding equation :

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \left(\sup_z \mathbb{P}^z \left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-B(n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(n) < k \le C(n) \right) \right)$$
$$(2.4) \le \quad -C^{\beta,\gamma}_{L_1, L_2 + \frac{L_2(\beta)-L_1(\beta)}{N}} \le -C^{\beta,\gamma}_{L_1, L_2+\varepsilon},$$

where the $\sup_z$ is over the $z \in \mathbb{R}$ such that $x_n L_1^n(B(n)) \le z \le x_n L_2^n(B(n))$.
In the computation of (2.4), we used the fact, obvious from its definition, that $C^{\beta,\gamma}_{L_1,L_2}$ only depends on $L_2 - L_1$, β and γ, which implies

$$\forall 0 \le j \le N-1, C^{\beta,\gamma}_{L_1 - \frac{L_1^n(\beta)(N-j-1)+L_2^n(\beta)(j+1)}{N}, L_2 - \frac{L_1^n(\beta)(N-j)+L_2^n(\beta)j}{N}} = C^{\beta,\gamma}_{L_1, L_2 + \frac{L_2(\beta)-L_1(\beta)}{N}}.$$

We make $\varepsilon \to 0$, then $C^{\beta,\gamma}_{L_1,L_2+\varepsilon} \to C^{\beta,\gamma}_{L_1,L_2}$ and we conclude using the bound (2.4). $\square$

Proposition 2.9. *Set L_1, $L_2 \in \mathcal{C}[0,1]$, with $L_1 < L_2$ and $L_1(0) < 0 < L_2(0)$. Let $(L_1^n)_n$ and $(L_2^n)_n$ be sequences of $\mathcal{F}[0,1]$ such that $\|L_1^n - L_1\|_\infty \to 0$ and $\|L_2^n - L_2\|_\infty \to 0$ as $n \to \infty$. We assume B and C are mappings $[0,1] \times \mathbb{N}^* \mapsto \mathbb{N}^*$, nondecreasing in the first component and such that, for any $\alpha \in [0,1]$, the sequences $(B(\alpha,n) - \alpha n)_n$ and $(C(\alpha,n) - \alpha n)_n$ are bounded.*

We know from Lemma 2.8 that for any $0 \le \beta < \gamma \le 1$, we have

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \left(\sup_z \mathbb{P}^z \left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-B(\beta,n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(\beta,n) < k \le C(\gamma,n) \right) \right)$$
$$\le -C^{\beta,\gamma}_{L_1,L_2},$$

where the $\sup_z$ is over the $z \in \mathbb{R}$ such that $x_n L_1^n(B(\beta,n)) \le z \le x_n L_2^n(B(\beta,n))$.

The claim of the present proposition is that this estimate holds uniformly in β and γ such that $0 \le \beta < \gamma \le 1$.

Proof. Let $\varepsilon > 0$. Let N be an integer such that

$$\forall 0 \leq \beta \leq \gamma \leq 1, (\gamma - \beta < \frac{1}{N} \Rightarrow C^{\beta,\gamma}_{L_1,L_2} < \varepsilon).$$

We apply Lemma 2.8 $N(N+1)/2$ times with

$$\beta = \frac{b}{N}, \quad \gamma = \frac{c}{N}, \quad 0 \leq b < c \leq N.$$

Then for n big enough, and any integers b and c such that $0 < b \leq c < N$, we have

$$\begin{aligned}
&\frac{1}{n^{1/3}} \log\left(\sup_z \mathbb{P}^z\left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-B(\frac{b}{N},n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(\frac{b}{N},n) < k \leq C(\frac{c}{N},n)\right)\right) \\
&\leq -C^{\frac{b}{N},\frac{c}{N}}_{L_1,L_2} + \varepsilon,
\end{aligned}$$

where the $\sup_z$ is over the $z \in \mathbb{R}$ such that $x_n L_1^n(B(\beta,n)) \leq z \leq x_n L_2^n(B(\beta,n))$.
For any $0 \leq \beta \leq \gamma \leq 1$, we can find $1 \leq b \leq N$ and $0 \leq c \leq N-1$ such that :

$$\frac{b-1}{N} \leq \beta \leq \frac{b}{N}, \quad \frac{c}{N} \leq \gamma \leq \frac{c+1}{N}.$$

If $b \leq c$, then

$$\begin{aligned}
&\frac{1}{n^{1/3}} \log\left(\sup_z \mathbb{P}\left(L_1^n\left(\frac{k}{n}\right) < \frac{z + S_{k-B(\beta,n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(\beta,n) < k \leq C(\gamma,n)\right)\right) \\
\leq\ &\frac{1}{n^{1/3}} \log\left(\sup_z \mathbb{P}\left(L_1^n\left(\frac{k}{n}\right) < \frac{z + S_{k-B(\frac{b}{N},n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(\frac{b}{N},n) < k \leq -C(\frac{c}{N},n)\right)\right) \\
\leq\ &-C^{\frac{b}{N},\frac{c}{N}}_{L_1,L_2} + \varepsilon \\
\leq\ &-C^{\beta,\gamma}_{L_1,L_2} + 3\varepsilon,
\end{aligned}$$

where the $\sup_z$ are over the $z \in \mathbb{R}$ such that $x_n L_1^n(B(\beta,n)) \leq z \leq x_n L_2^n(B(\beta,n))$.
Else $b = c+1$, $\gamma - \beta \leq 1/N$, hence $C^{\beta,\gamma}_{L_1,L_2} < \varepsilon$. This case is easier :

$$\begin{aligned}
&\frac{1}{n^{1/3}} \log\left(\sup_z \mathbb{P}^z\left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-B(\beta,n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(\beta,n) < k \leq C(\gamma,n)\right)\right) \\
\leq\ &0 \leq -C^{\beta,\gamma}_{L_1,L_2} + \varepsilon.
\end{aligned}$$

□

Remark 2.10. The upper bound above is sharp and may be replaced by an equivalence. Keeping the above notations and hypothesis, we have, for any $\varepsilon > 0$ small enough,

$$\log\left(\inf_z \mathbb{P}^z\left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-B(n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(n) < k \leq C(n)\right)\right) \sim -nx_n^{-2}C^{\beta,\gamma}_{L_1,L_2},$$

where the $\inf_z$ is over the $z \in \mathbb{R}$ such that $x_n\left(L_1^n(B(\beta,n)) + \varepsilon\right) < z < x_n\left(L_2^n(B(\beta,n)) - \varepsilon\right)$. The proof of this result is very similar to the upperbound, but since it is not useful here, we omit it.

Corollary 2.11. *The upper bound in Theorem 2.3 is still valid with condition (2.1) replaced by*

$$g_1(0) \leq 0 \leq g_2(0) \text{ and } \forall t \in [0,1], g_1(t) \leq g_2(t). \tag{2.5}$$

Proof. Let g_1, g_2 be like in the statement of the corollary. For any $\varepsilon > 0$, replace g_1 by $\widetilde{g}_1 := g_1 - \varepsilon$ and g_2 by $\widetilde{g}_2 := g_2 + \varepsilon$.

We apply Theorem 2.3 with the functions $\widetilde{g}_1$ and $\widetilde{g}_2$:

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbb{P}(\widetilde{g}_1 <_n s_n <_n \widetilde{g}_2) \leq -\frac{\pi^2\sigma^2}{2}\int_0^1 \frac{\mathrm{d}t}{[\widetilde{g}_2(t) - \widetilde{g}_1(t)]^2}.$$

Using the fact that $\{g_1 <_n s_n <_n g_2\} \subset \{\widetilde{g}_1 <_n s_n <_n \widetilde{g}_2\}$, we obtain

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbb{P}(g_1 <_n s_n <_n g_2) \leq -\frac{\pi^2\sigma^2}{2}\int_0^1 \frac{\mathrm{d}t}{[2\varepsilon + g_2(t) - g_1(t)]^2}.$$

We make ε tend to 0 and conclude with the monotone convergence theorem. □

Remark 2.12. For the upper bounds in the preceding results and in particular in Proposition 2.9, we can release the hypothesis (2.1) to (2.5), thanks to the argument we used in Corollary 2.11.

When estimating the expectations that appear in Lemma 2.1, we need some information about the position of the walker at the end of his journey between the barriers we consider. Thee following result tells us that the main contribution comes from the individuals close to $g_2(1)x_n$:

Corollary 2.13. *Set L_1, $L_2 \in \mathcal{C}[0,1]$, with $L_1 < L_2$ and $L_1(0) < 0 < L_2(0)$. Let $(L_1^n)_n$ and $(L_2^n)_n$ be sequences of $\mathcal{F}[0,1]$ such that $\|L_1^n - L_1\|_\infty \to 0$ and $\|L_2^n - L_2\|_\infty \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Then*

$$\log\left(\mathbb{P}(L_1^n <_n s_n <_n L_2^n, s_n(1) > L_2^n(1) - \varepsilon)\right) \sim -nx_n^{-2}C_{L_1,L_2},$$

Proof. It clearly suffices to prove the lower bound. Set $\eta > 0$ such that $\eta < \inf_{n\geq 1} \|L_2^n - L_1^n\|_\infty$. Obviously, we also have $\eta < \|L_2 - L_1\|_\infty = \lim_n \|L_2^n - L_1^n\|_\infty$.

For any $A > 0$, we define the sequence of functions $(\widetilde{L}_1^{A,n})_n$ by :

$$\forall n \geq 1, \forall 0 \leq t \leq 1, \widetilde{L}_1^{A,n}(t) := \max(L_1^n(t), \min(L_2^n(t) - \eta, L_2^n(1) - \varepsilon - A(1-t))).$$

As $n \to \infty$, $\|\widetilde{L}_1^{A,n} - \widetilde{L}_1^A\|_\infty \to 0$, where

$$\forall 0 \leq t \leq 1, \widetilde{L}_1^A(t) := \max(L_1(t), \min(L_2(t) - \eta, L_2(1) - \varepsilon - A(1-t))).$$

$\widetilde{L}_1^A$ is continuous on $[0,1]$ and when $A \to \infty$, we have for any $0 \leq t < 1$:

$$\widetilde{L}_1^A(t) \to \max(L_1(t), L_2(t) - \eta) = L_1(t);$$

whence $C_{\widetilde{L}_1^A, L_2} \to C_{L_1, L_2}$.

For any $A > 0$, we have $\widetilde{L}_1^{A,n} \leq L_1^n$, and $\widetilde{L}_1^{A,n}(1) > L_2^n(1) - \varepsilon$ whence

$$\mathbb{P}(L_1^n <_n s_n <_n L_2^n, s_n(1) > L_2^n(1) - \varepsilon) \geq \mathbb{P}(L_1^n <_n s_n <_n \widetilde{L}_2^{A,n}). \tag{2.6}$$

We assume that A is large enough to have $\widetilde{L}_1^A(0) \leq L_1(0) < 0$. Besides $\forall n \geq 1$, $\widetilde{L}_1^{A,n} \leq \max(L_1^n, L_2^n - \eta) = L_2^n - \eta$. Therefore we are allowed to apply Lemma 2.6 to estimate the right-hand side of (2.6). It follows that

$$\liminf_{n\to\infty} \frac{x_n^2}{n} \log \mathbb{P}(L_1^n <_n s_n <_n L_2^n, s_n(1) > L_2^n(1) - \varepsilon) \geq -C_{\widetilde{L}_1^A, L_2}.$$

We conclude by letting $A \to \infty$. □

The following generalization of Theorem 2.3 is useful to deal with Galton-Watson trees of infinite degree ; we refer to Gantert, Hu and Shi [44] (Lemma 2.1) for the proof.

Lemma 2.14 (Triangular version of Mogul'skii's estimate)**.** *For each $n \geq 1$, let $X_i^{(n)}$, $1 \leq i \leq n$ be i.i.d. real-valued random variables. We define $S_i^{(n)} = S_0^{(n)} + X_1^{(n)} + \cdots + X_i^{(n)}$ for $1 \leq i \leq n$, we write the initial value of this random walk in exponent of the probability operator : for any $x \in \mathbb{R}$, $\mathbb{P}^x(S_0^{(n)} = x) = 1$, and we simply write $\mathbb{P} = \mathbb{P}^0$ when the walks starts from the origin. We define*

$$s_n(t) := \frac{S_{\lfloor tn \rfloor}^{(n)}}{x_n} = \frac{\xi_1 + \xi_2 + \ldots + \xi_k}{x_n} \qquad \textit{for} \qquad k/n \leq t < (k+1)/n\,.$$

Assume that there exist constants $\delta > 0$ and $\sigma^2 > 0$ such that

$$\text{(2.7)} \qquad \sup_{n\geq 1} \mathbb{E}\left[\left|X_1^{(n)}\right|^{2+\delta}\right] < +\infty, \quad \mathbb{E}\left[X_1^{(n)}\right] = o\left(n^{-2/3}\right) \text{ and } Var\left(X_1^{(n)}\right) \to \sigma^2.$$

Let g_1 and g_2 be continuous functions $[0,1] \to \mathbb{R}$ such that

$$g_1(0) < 0 < g_2(0) \text{ and } g_1 < g_2.$$

Then

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbb{P}(g_1 <_n s_n <_n g_2) = -C_{g_1,g_2}.$$

In Section 4, we need a lower bound in a particular case with $g_1(0) < 0 = g_2(0)$ and $g_2 \geq 0$ on $[0,1]$. In order to apply the results above, the following lemma will be useful :

Lemma 2.15. *There are constants $M \geq 1$, and $\varepsilon_1 > 0$ such that, with $k := \lfloor \varepsilon_2 n^{1/3} \rfloor$, such that the probability $P_n(M, \varepsilon_1, \varepsilon_2)$ defined as*

$$\mathbb{P}\left(\exists u \in \mathcal{T}_k, \forall i < k, \#\Gamma(u_i) \leq M, g_1\left(\frac{i}{n}\right) \leq \frac{V(u_i)}{n^{1/3}} \leq g_2\left(\frac{i}{n}\right); -M\varepsilon_2 \leq \frac{V(u_k)}{n^{1/3}} \leq -\varepsilon_1\varepsilon_2\right)$$

satisfies

$$\lim_{\varepsilon_2\to 0} \liminf_{n\to\infty} \frac{1}{n^{1/3}} \log P_n(M, \varepsilon_1, \varepsilon_2) = 0.$$

Proof. Let $\varepsilon_1 > 0$ and such that for some $M \geq 1$,

$$p := \mathbb{P}\left(\#\mathcal{T}_1 \leq M; \exists u \in \mathcal{T}_1, -M \leq V(u) \leq -\varepsilon_1\right) > 0.$$

By independence,

$$\mathbb{P}\left(\exists u \in \mathcal{T}_k, \forall i < k, \#\Gamma(u_i) \leq M, \forall i \leq k, -iM \leq V(u_i) \leq -i\varepsilon_1\right) \geq p^k.$$

Let $\varepsilon_2 > 0$ such that $M\varepsilon_2 < -g_1(0)$. For any integer n large enough, we take $k := \lfloor \varepsilon_2 n^{1/3} \rfloor$. Hence, for $\varepsilon_2 > 0$ small enough, we have

$$\mathbb{P}\left(\exists u \in \mathcal{T}_k, \forall i < k, \#\Gamma(u_i) \leq M; g_1\left(\frac{i}{n}\right) \leq \frac{V(u_i)}{n^{1/3}} \leq g_2\left(\frac{i}{n}\right); -M\varepsilon_2 \leq \frac{V(u_k)}{n^{1/3}} \leq -\varepsilon_1\varepsilon_2\right) \geq p^k.$$

$\square$

We shall use Lemma 2.15 combined the following variant of Mogul'skii's result. We do not write a complete proof, since we obtain this result by starting with the triangular version (Lemma 2.14), then by applying arguments analogous to the ones of Lemma 2.8 (but for the lower bound) and finally by applying the argument of Corollary 2.13.

Proposition 2.16. *Let, for $0 \leq i \leq n$, $X_i^{(n)}$, $S_i^{(n)}$, and s_n be like in Lemma 2.14.*

Let g_1, g_2, $(g_1^n)_n$, $(g_2^n)_n$, β, γ, B and C be like in Lemma 2.8. Let u^* and v^* be real numbers such that $g_1(\beta) < u^* < v^* < g_2(\beta)$. Let u_n and v_n be sequences of real numbers such that

$$\frac{u_n}{x_n} \to u^*, \quad \frac{v_n}{x_n} \to v^*, \quad L_1^n(B(n))x_n \leq u_n \leq v_n \leq L_2^n(B(n))x_n \, \forall n \geq 1.$$

We have, for any $\varepsilon > 0$,

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log \left(\inf_z \mathbb{P}^z \left(\forall B(n) < k \leq C(n), g_1^n\left(\frac{k}{n}\right) < \frac{S_{k-B(n)}^{(n)}}{n^{1/3}} < g_2^n\left(\frac{k}{n}\right); \right.\right.$$
$$\left.\left. g_2^n\left(\frac{C(n)}{n}\right) - \varepsilon < \frac{S_{C(n)-B(n)}^{(n)}}{n^{1/3}} < g_2^n\left(\frac{C(n)}{n}\right) \right)\right) \geq -C_{L_1,L_2}^{\beta,\gamma}$$

where the $\inf_z$ is over the $z \in \mathbb{R}$ such that $u_n \leq z \leq v_n$.

3 Upper bound for the survival probability

3.1 Splitting the survival probability

Fix $a > 0$. Obviously,

$$\mathbb{P}\left(\exists u \in \mathcal{T}_\infty, \forall i, V(u_i) \leq a i^{1/3}\right) = \lim_{n\to\infty} \mathbb{P}\left(\exists u \in \mathcal{T}_n, \forall i \leq n, V(u_i) \leq a i^{1/3}\right).$$

From now on, $n \geq 1$ is fixed.

We set a second barrier $i \mapsto a i^{1/3} - b_{i,n}$ (with $b_{i,n} > 0$ for $1 \leq i \leq n$ yet to be determined) below the first one $i \mapsto a i^{1/3}$: if a particle crosses it, then its descendants will be likely to stay below the first one until generation n.

Let $H(u) := \inf\{k \leq n : V(u_k) < a k^{1/3} - b_{k,n}\}$ be the first time the line of descent of a particle $u \in \mathcal{T}_n$ crosses this second barrier ($H(u) = \infty$ if the particle stays between the barriers until time n). We split the sum accordingly :

$$\mathbb{P}\left(\exists u \in \mathcal{T}_n, \forall i \leq n, V(u_i) \leq a i^{1/3}\right) \leq R_\infty + \sum_{j=1}^{n} R_j, \tag{3.1}$$

where

$$R_j = \mathbb{P}\left(\exists u \in \mathcal{T}_n, H(u) = j, \forall i \leq n, V(u_i) \leq a i^{1/3}\right) \text{ for } j = 1, \ldots, n, \infty.$$

By Chebyshev's inequality and then Lemma 2.1, we get

$$\begin{aligned}
R_\infty &\leq \mathbb{E}\left[\sum_{u\in\mathcal{T}_n} \mathbb{1}_{\left\{\forall i\leq n,\, ai^{1/3}-b_{i,n}\leq V(u_i)\leq ai^{1/3}\right\}}\right] \\
&\leq \mathbb{E}\left[\mathrm{e}^{S_n}\mathbb{1}_{\left\{\forall i\leq n,\, ai^{1/3}-b_{i,n}\leq S_i\leq ai^{1/3}\right\}}\right] \\
&\leq \mathrm{e}^{an^{1/3}}\mathbb{P}\left(\forall i\leq n,\, ai^{1/3}-b_{i,n}\leq S_i\leq ai^{1/3}\right). \qquad (3.2)
\end{aligned}$$

For $1\leq j\leq n$,

$$\begin{aligned}
R_j &\leq \mathbb{E}\left[\sum_{v\in\mathcal{T}_j} \mathbb{1}_{\left\{\forall i<j,\, ai^{1/3}-b_{i,n}\leq V(v_i)\leq ai^{1/3}, V(v)<aj^{1/3}-b_{j,n}\right\}}\right] \\
&\leq \mathbb{E}\left[\mathrm{e}^{S_j}\mathbb{1}_{\left\{\forall i<j,\, ai^{1/3}-b_{i,n}\leq S_i\leq ai^{1/3},\, V(S_j)<aj^{1/3}-b_{j,n}\right\}}\right] \\
&\leq \mathrm{e}^{aj^{1/3}-b_{j,n}}\mathbb{P}\left(\forall i<j,\, ai^{1/3}-b_{i,n}\leq S_i\leq ai^{1/3}\right). \qquad (3.3)
\end{aligned}$$

3.2 Asymptotics for R_∞

In order to apply Corollary 2.11, we set $b_{i,n} := n^{1/3}g(\frac{i}{n})$ for some continuous function $g:[0,1]\mapsto[0,+\infty)$. We take for any $t\in[0,1]$, $g_2(t):=at^{1/3}$ and $g_1(t)=g_2(t)-g(t)$. Then we have

$$\limsup_{n\to\infty}\frac{\log\mathbb{P}\left(\forall i\leq n, ai^{1/3}-b_{i,n}\leq S_i\leq ai^{1/3}\right)}{n^{1/3}}\leq -C_{g_1,g_2}. \qquad (3.4)$$

Putting together equations (3.2) and (3.4), we get

$$\limsup_{n\to\infty}\frac{\log R_\infty}{n^{1/3}}\leq -s_1, \qquad (3.5)$$

where

$$s_1 := -a + C_{g_1,g_2} = -a+\frac{\pi^2\sigma^2}{2}\int_0^1\frac{\mathrm{d}t}{g(t)^2}. \qquad (3.6)$$

3.3 Asymptotics for R_j

We define B and C by $B(\alpha,n):=0$ and $C(\alpha,n):=\lfloor\alpha n\rfloor+1$ and write, for any $\alpha\in(0,1)$, $j:=C(\alpha,n)$. Proposition 2.9 (combined with Remark 2.12) yields that, uniformly in $\alpha\in(0,1)$,

$$\limsup_{n\to\infty}\frac{1}{n^{1/3}}\log\mathbb{P}\left(\forall i<j, ai^{1/3}-n^{1/3}g(\frac{i}{n})\leq S_i\leq ai^{1/3}\right)\leq -C^{0,\alpha}_{g_1,g_2}.$$

Putting together equations (3.3) and (3.3), we get that, uniformly in $\alpha \in (0,1)$,

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log R_j \leq a\alpha^{1/3} - g(\alpha) - C^{0,\alpha}_{g_1,g_2}. \tag{3.7}$$

Obviously, for any $n \geq 1$,

$$\sum_{j=1}^{n} R_j(n) \leq n \sup_{1\leq j\leq n} R_j(n) = n \sup_{0<\alpha<1} R_{C(\alpha,n)}(n). \tag{3.8}$$

As a consequence,

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \sum_{j=1}^{n} R_j(n) \leq -s_2. \tag{3.9}$$

where

$$s_2 := \min_{0\leq\alpha\leq 1} \left\{-a\alpha^{1/3} + g(\alpha) + C^{0,\alpha}_{g_1,g_2}\right\} = \min_{0\leq\alpha\leq 1} \left\{-a\alpha^{1/3} + g(\alpha) + \frac{\pi^2\sigma^2}{2}\int_0^\alpha \frac{\mathrm{d}t}{g(t)^2}\right\}. \tag{3.10}$$

Combining (3.9) with (3.5) and (3.1), we obtain

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\left(\exists u \in \mathcal{T}_n, \forall i \leq n, V(u_i) \leq a i^{1/3}\right) \leq -s,$$

where $s := \min(s_1, s_2)$.

3.4 Choice of g for the upper bound

Set $a \in (0, a_c)$. We are looking for a function g such that $s > 0$. The existence of such a function implies extinction and ends the proof the first part of Theorem 1.2.

We add the constraint $g(1) = 0$, (but assume $\int_0^1 \frac{\mathrm{d}u}{g(u)^2} < \infty$). Taking $\alpha = 1$, we see from (3.10) and (3.6) that this implies $s_2 \leq s_1$ and, as a result, $s = s_2$.

We choose g in such a way that the quantity $-a\alpha^{1/3} + g(\alpha) + \frac{\pi^2\sigma^2}{2}\int_0^\alpha \frac{\mathrm{d}t}{g(t)^2}$ which appears in (3.10) does not depend on α. Hence g is defined as the solution of the equation :

$$\forall t \in [0,1], -at^{1/3} + f(t) + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{\mathrm{d}u}{f(u)^2} = s, \tag{3.11}$$

where s is some positive constant, the value of which is to be set later in such a way that $f(1-) = 0$. According to the computations above, this value of s will give a bound for the rate of decay of the survival probability.

Equivalently, equation (3.11) may be written $f(0) = s$ and $\forall t \in (0,1)$,

$$f'(t) = \frac{a}{3}t^{-2/3} - \frac{\pi^2\sigma^2}{2f(t)^2}. \tag{3.12}$$

By the Picard-Lindelöf theorem (see for example [5]), such an ordinary differential equation admits a unique maximal solution f defined on an interval $[0, t_{\max})$ with $t_{\max} \in (0, +\infty]$. And if $t_{\max} < +\infty$, then f has limit 0 or $+\infty$ when t goes to $t_{\max}$.

Remark 3.1. The fact that $f'(0)$ does not exist here is not troublesome at all since the proof of the theorem, using Picard iterates, actually relies on equation (3.11).

In order to prove that there exists an initial value s such that $t_{\max} = 1$ and $\lim_{t\to 1} f(t) = 0$, we get a closer look at the differential equation.

First we state three simple results specific to this differential equation.

Proposition 3.2. *Let $\lambda > 0$ and f a continuous function $[0, t_0) \mapsto (0, +\infty)$. Define f_λ : $(0, \lambda^{-1}t_0) \mapsto (0, +\infty)$ by*

$$f_\lambda(t) = \lambda^{-1/3} f(\lambda t).$$

Then f satisfies equation (3.12) on $(0, t_0)$ if and only if f_λ does on $(0, \lambda^{-1}t_0)$.

Proof. Assume that f satisfies equation (3.12) for any $0 < t < t_0$. Then for any $0 < t < \lambda^{-1}t_0$,

$$\begin{aligned} f'_\lambda(t) &= \lambda^{2/3} f'(\lambda t) \\ &= \lambda^{2/3}\left(\frac{a}{3}(\lambda t)^{-2/3} - \frac{\pi^2\sigma^2}{2f(\lambda t)^2}\right) \\ &= \frac{a}{3}t^{-2/3} - \frac{\pi^2\sigma^2}{2f_\lambda(t)^2}. \end{aligned}$$

This means that f_λ also satisfies equation (3.12) for any $0 < t < \lambda^{-1}t_0$.

Conversely, assume that f_λ satisfies equation (3.12) on $(0, \lambda^{-1}t_0)$. We notice that if $\lambda' > 0$, then $(f_\lambda)_{\lambda'} = f_{\lambda\lambda'}$. We take $\lambda' = \lambda^{-1}$. Hence $(f_\lambda)_{\lambda'} = f$ also satifies equation (3.12) for any $0 < t < (\lambda\lambda')^{-1}t_0 = t_0$. □

Proposition 3.3. *Set $0 < a_1 < a_2$ and $s > 0$. Let f_1 and f_2 be functions $[0, t_{\max}) \mapsto (0, +\infty)$ such that*

$$\forall 0 \le t < t_{\max}, \forall i \in \{1,2\},\ -at^{1/3} + f_i(t) + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{\mathrm{d}u}{f_i(u)^2} = s.$$

Then, for all $0 \le t < t_{\max}$, $f_1(t) \le f_2(t)$.

Proof. It suffices to prove that, if $0 \leq t_{\text{start}}$, $0 < a_1 < a_2$ and $0 < x_1 \leq x_2$, then there exist $t_{\text{next}} > t_{\text{start}}$ such that there are functions f_1 and $f_2 : [t_{\text{start}}, t_{\text{next}}) \mapsto (0, +\infty)$ such that

$$\forall t_{\text{start}} \leq t < t_{\text{next}}, \forall i \in \{1,2\},\ -a_i(t^{1/3} - t_{\text{start}}^{1/3}) + f_i(t) + \frac{\pi^2\sigma^2}{2}\int_{t_{\text{start}}}^{t} \frac{\mathrm{d}u}{f_i(u)^2} = x_i;$$

then, for any $t_{\text{start}} \leq t < t_{\text{next}}$, $f_1(t) \leq f_2(t)$.

We choose t_{next} such that the Picard interates f_i^n defined, for $i \in \{1,2\}$, by :

$$\forall t_{\text{start}} \leq t < t_{\text{next}},\ f_i^0(t) = x_i;$$

$$\forall n \in \mathbb{N}, \forall t_{\text{start}} \leq t < t_{\text{next}}, f_i^{n+1}(t) = f_i^n(t_{\text{start}}) + a_i(t^{1/3} - t_{\text{start}}^{1/3}) - \frac{\pi^2\sigma^2}{2}\int_{t_{\text{start}}}^{t} \frac{\mathrm{d}u}{f_i^n(u)^2},$$

exist and converge on $[t_{\text{start}}, t_{\text{next}})$. The limits f_i are solutions of the integral equations for $i \in \{1,2\}$.

It is easy to prove by induction on n that

$$\forall n \in \mathbb{N}, \forall t_{\text{start}} \leq t < t_{\text{next}}, f_1^n(t) \leq f_2^n(t).$$

Letting n tend to infinity gives us the desired conclusion. $\square$

Proposition 3.4. *Let f be as above. Then we are in one of the following cases :*

(A) $t_{\max} = +\infty$ and $f(t) \to +\infty$ as $t \to +\infty$;

(B) $t_{\max} < +\infty$ and $f(t) \to 0$ as $t \to t_{\max}$.

Proof. First notice that for any $0 < t < t_{\max}$, $f(t) \leq s + at^{1/3}$. A consequence of this inequality is that if $t_{\max} < +\infty$, then the limit of f when t goes to $t_{\max}$ can only be 0.

Now, suppose that $t_{\max} = +\infty$ but that f does not go to infinity. Then there are $M > 0$ and a sequence $(t_n)_n \geq 1$ with $\lim_n t_n = +\infty$ such that for any $n \geq 1$, $f(t_n) \leq M$. We can choose n such that $\frac{a}{3}t_n^{-2/3} - \frac{\pi^2\sigma^2}{2M^2} < 0$.

Then it is easy to see that f decreases after t_n. Indeed, consider

$$t_* := \inf\left\{t \geq t_n, f'(t) > \frac{a}{3}t_n^{-2/3} - \frac{\pi^2\sigma^2}{2M^2}\right\}.$$

We have, for $t_n \leq t \leq t_*$,

$$f'(t) = \frac{a}{3}t^{-2/3} - \frac{\pi^2\sigma^2}{2f(t)^2} \leq \frac{a}{3}t_n^{-2/3} - \frac{\pi^2\sigma^2}{2M^2} < 0.$$

If we assume $t_* < +\infty$, then $f'(t_*) < 0$, then f decreases in a neighborhood of t_* and the inequality $f'(t) \leq \frac{a}{3}t_n^{-2/3} - \frac{\pi^2\sigma^2}{2M^2}$ still holds on this neighborhood, which contradicts the definition of t_*.

We have proved that $f'(t)$ is less than a negative constant for $t \geq t_n$, which implies that f reaches zero in finite time. $\square$

Assume we are in the second case of Proposition 3.4. We set $\lambda := t_{\max}^{-1}$ and define the function f_λ like in Proposition 3.2 (with $t_0 = t_{\max}$). We choose $g = f_\lambda$ and set $g(1) = 0$ so that g is continuous over $[0, 1]$ and satifies (3.12) for all $t \in (0, 1)$.

Remark 3.5. A consequence of Proposition 3.2 is that the choice of the value s of $f(0)$ does not matter at all. If we replace $s > 0$ with another $\widetilde{s} > 0$, we then replace λ with $\widetilde{\lambda} = \lambda\left(\frac{\widetilde{s}}{s}\right)^3$ and finally get the same g.

So we only have to prove that, when $a < a_c$, we are in case (B) of Proposition 3.4, and we will deduce the upper bound in Theorem 1.2. This is contained in the following :

Proposition 3.6. *Let f be the solution of equation (3.12) with initial condition $f(0) = 1$.*

(i) If $a > a_c$, then $t_{\max} = +\infty$ and $f(t) \sim bt^{1/3}$ as $t \to +\infty$ with b defined by $b > \frac{2a_c}{3}$ and $a = b + \frac{3\pi^2\sigma^2}{2b^2}$.

(ii) If $a = a_c$, then $t_{\max} = +\infty$ and $f(t) \sim \frac{2a_c}{3}t^{1/3}$ as $t \to +\infty$.

(iii) If $a < a_c$, then $t_{\max} < +\infty$ and $f(t) \to 0$ as $t \to t_{\max}$.

In the proof of the proposition, we will need the following lemma :

Lemma 3.7. *Assume that f is a solution on $[0, +\infty)$ of the differential equation and that :*

$$\limsup_{t\to+\infty} \frac{f(t)}{t^{1/3}} \leq b.$$

Then we have

$$\limsup_{t\to+\infty} \frac{f(t)}{t^{1/3}} \leq b' := a - \frac{3\pi^2\sigma^2}{2b^2}.$$

Proof of Lemma 3.7. Let $\varepsilon > 0$. By hypothesis, for any t greater than some t_0, we have $f(t) \leq (b+\varepsilon)t^{1/3}$. For some real constants c_0 and c_0' and any $t \geq t_0$, we have, by equation (3.11) :

$$f(t) \leq c_0 + at^{1/3} - \frac{\pi^2\sigma^2}{2(b+\varepsilon)^2}\int_{t_0}^{t} \frac{\mathrm{d}u}{u^{2/3}} = c_0' + \left(a - \frac{3\pi^2\sigma^2}{2(b+\varepsilon)^2}\right)t^{1/3}.$$

Hence
$$\limsup_{t\to+\infty} \frac{f(t)}{t^{1/3}} \leq \left(a - \frac{3\pi^2\sigma^2}{2(b+\varepsilon)^2}\right).$$
Letting ε tend to 0 ends the proof of the lemma. □

Iterating Lemma 3.7, we obtain :

Lemma 3.8. *Assume that f is a solution on $[0,+\infty)$ of the differential equation and let b_0 be a real such that $b_0 \geq \limsup_{t\to+\infty} \frac{f(t)}{t^{1/3}}$. We define the sequence $(b_n)_{n\in\mathbb{N}}$ recursively by $b_{n+1} := a - \frac{3\pi^2\sigma^2}{2b_n^2}$. Then*
$$\forall n \geq 1, \limsup_{t\to+\infty} \frac{f(t)}{t^{1/3}} \leq b_n.$$

Proof of Proposition 3.6. (i) Assume $a \geq a_c$ and let b such that $a = b + \frac{3\pi^2\sigma^2}{2b^2}$. Define, for $0 \leq t \leq t_{\max}$, $f_0(t) := bt^{1/3}$. Then f_0 satisfies equation (3.12) as f does, with initial condition $f_0(0) = 0 < f(0) = s$. Hence
$$\forall 0 \leq t \leq t_{\max}, f(t) \geq f_0(t).$$

This implies $t_{\max} = +\infty$. Now let $h = f - f_0$. Then, by equation (3.11), we have, for $t \geq 0$,
$$\begin{aligned} h(t) &= s + (a-b)t^{1/3} - \int_0^t \frac{\pi^2\sigma^2\,\mathrm{d}u}{2f(u)^2} \\ &= s + (a - b - \frac{3\pi^2\sigma^2}{2b^2})t^{1/3} + \int_0^t \frac{\pi^2\sigma^2\,\mathrm{d}u}{2}\left(\frac{1}{f_0(u)^2} - \frac{1}{f(u)^2}\right). \end{aligned}$$
Since $a = b + \frac{3\pi^2\sigma^2}{2b^2}$,
$$h(t) = s + \int_0^t \frac{\pi^2\sigma^2\,\mathrm{d}u}{2}\left(\frac{1}{f_0(u)^2} - \frac{1}{f(u)^2}\right) \leq s + \int_0^t \frac{\pi^2\sigma^2}{2}\frac{2h(u)\,\mathrm{d}u}{f_0(u)^3}.$$

We apply Gronwall's lemma and obtain, for any $0 < t_0 < t$,

$$h(t) \leq h(t_0)\exp\left(\int_{t_0}^t \frac{\pi^2\sigma^2\,\mathrm{d}u}{b^3u}\right) = h(t_0)\left(\frac{t}{t_0}\right)^{\frac{\pi^2\sigma^2}{b^3}}. \tag{3.13}$$

Notice that $\frac{\pi^2\sigma^2}{b^3} = \frac{1}{3}\left(\frac{2a_c}{3b}\right)^3$. Then if $a > a_c$ and $b > \frac{2a_c}{3}$, the exponent in the right-hand side of (3.13) will be less than $\frac{1}{3}$. Hence inequality (3.13) implies (i).

(ii) Assume $a = a_c$ and $b = \frac{2a_c}{3}$. This is the same as when $a > a_c$, except that the exponent in the right-hand side of (3.13) is exactly $\frac{1}{3}$, which means that for some constant $b_0 > \frac{2a_c}{3}$,
$$\forall t \geq t_0,\ f_0(t) \leq f(t) \leq b_0 t^{1/3}.$$

Apply Lemma 3.8. The result follows from that $\lim_n b_n = \frac{2a_c}{3}$.

(iii) Assume $a < a_c$ and $t_{\max} = +\infty$. Then, by (ii) and Proposition 3.3, we have that any $b_0 > \frac{2a_c}{3}$, for t large enough,

$$\limsup_{t\to+\infty} \frac{f(t)}{t^{1/3}} \leq b_0.$$

We apply Lemma 3.8. If b_0 is close enough to $\frac{2a_c}{3}$, we will have $b_1 < \frac{2a_c}{3}$ and $b_n \to -\infty$ as n goes to infinity, which is absurd. We conclude that the hypothesis $t_{\max} = +\infty$ is false, which proves the proposition. □

4 Lower bound for the survival probability

4.1 Strategy of the estimate

The basic idea is to consider only the population between two barriers (below $i \mapsto ai^{1/3}$ but above $i \mapsto (a-b)i^{1/3}$), estimate the first two moments of the number of individuals in generation n and then to use the Paley-Zygmund inequality to get the lower bound.

Unfortunately, Mogul'skii's estimate causes the appearance of a factor $e^{o(n^{1/3})}$ in the estimates of the moments of the surviving population at generation n, so we will not be able to prove directly that the population survives with positive probability.

Here is how to overcome this difficulty :

Set $\lambda > 0$ such that $e^\lambda \in \mathbb{N}$ and $(v_k)_{k\geq 1}$ a sequence of positive integers. We consider the population surviving below the barrier $i \mapsto ai^{1/3}$: any individual that would be born above this barrier is removed and consequently does not reproduce. For any $k \in \mathbb{N}$, we pick a single individual z at position $V(z)$ in generation $e^{\lambda k}$ and consider the number $Y_k(z)$ of descendants she eventually has in generation $e^{\lambda(k+1)}$.

We get a lower bound for $Y_k(z)$ by considering, instead of z, a virtual individual $\widetilde{z}$ in the same generation $e^{\lambda k}$ but positioned on the barrier at $\widetilde{V}(z) := ae^{\lambda k/3} \geq V(z)$. The number and displacements of the descendants of $\widetilde{z}$ are exactly the same as those of z. Obviously, for any $u > z$, $V(\widetilde{u}) = V(u) + ae^{\lambda k/3} - V(z) \geq V(u)$. Hence the descendants of $\widetilde{z}$ are more likely to cross the barrier and be killed, which means that $Y_k(\widetilde{z}) \leq Y_k(z)$.

In order to apply Mogul'skii's estimate, we add a second absorbing barrier $i \mapsto (a-b)i^{1/3}$ for some $b > 0$ and kill any descendant of $\widetilde{z}$ that is born below it. This way, we obtain that, almost surely, $Z_k \leq Y_k(\widetilde{z}) \leq Y_k(z)$, where

$$Z_k := \# \left\{ u \in \mathcal{T}_{e^{\lambda(k+1)}} : u > z, \forall e^{\lambda k} < i \leq e^{\lambda(k+1)}, (a-b)i^{1/3} \leq V(\widetilde{u}_i) \leq ai^{1/3} \right\}.$$

Clearly, Z_k depends on z but its law and in particular $A_k := \mathbb{P}(Z_k \geq v_k) \leq \mathbb{P}(Y_k(z) \geq v_k)$ do not.

We define, for any $n \geq 1$:

$$\mathcal{P}_n := \mathbb{P}\left(\forall 1 \leq k \leq n, \#\left\{u \in \mathcal{T}_{\mathrm{e}^{\lambda k}} : \forall i \leq \mathrm{e}^{\lambda k}, V(u_i) \leq a i^{1/3}\right\} \geq v_{k-1}\right).$$

If $1 \leq n_0 \leq n$, then we have :

$$\mathcal{P}_{n+1} \geq \mathcal{P}_n \left(1 - (1 - A_n)^{v_{n-1}}\right).$$

By induction, we obtain :

$$\mathcal{P}_n \geq \mathcal{P}_{n_0} \prod_{k=n_0}^{n-1} \left(1 - (1 - A_k)^{v_{k-1}}\right) \geq \mathcal{P}_{n_0} \prod_{k=n_0}^{n} \left(1 - \mathrm{e}^{-v_{k-1} A_k}\right).$$

$$\log \mathcal{P}_n \geq \log \mathcal{P}_{n_0} + \sum_{k=n_0}^{n} \log\left(1 - \mathrm{e}^{-v_{k-1} A_k}\right).$$

With the equivalent $\log(1+x) \sim x$ for small values of x, the previous inequality makes Proposition 1.4 a consequence of the following lemma :

Lemma 4.1. *If $a > a_c$, we can choose $(v_k)_{k \in \mathbb{N}}$ such that, when λ is large enough and such that $\mathrm{e}^{\lambda} \in \mathbb{N}$, we have*

$$\sum_{k=0}^{\infty} \mathrm{e}^{-v_k A_{k+1}} < +\infty. \tag{4.1}$$

Fix $\theta \in (0,1)$, for example $\theta = \frac{1}{2}$. The Paley-Zygmund inequality, with $v_k := \theta \mathbb{E}[Z_k]$ will provide us with the lower bound on A_k needed to prove Lemma 4.1 :

$$A_k \geq (1-\theta)^2 \frac{(\mathbb{E}[Z_k])^2}{\mathbb{E}[Z_k^2]}. \tag{4.2}$$

We set $k \geq 0$ and consider, as stated above, the descendants of an individual $\widetilde{z}$ starting at time $\mathrm{e}^{\lambda k}$ at position $a \mathrm{e}^{\lambda k/3}$ over $\ell_k := \mathrm{e}^{\lambda(k+1)} - \mathrm{e}^{\lambda k}$ generations. The individuals of generation i are killed and have no descendant if they are out of the interval :

$$I_i := \left[(a-b) i^{1/3}, a i^{1/3}\right].$$

We set, for $k = 0$ for example (then the equations also hold for all $k \in \mathbb{N}$ with the same functions) :

$$g_2(t) := a\left(\left(t + \frac{\mathrm{e}^{\lambda k}}{\ell_k}\right)^{1/3} - \left(\frac{\mathrm{e}^{\lambda k}}{\ell_k}\right)^{1/3}\right),\ g(t) := b\left(t + \frac{\mathrm{e}^{\lambda k}}{\ell_k}\right)^{1/3},\ g_1(t) := g_2(t) - g(t). \tag{4.3}$$

4.2 Upper bound for the second moment

We split the double sum over $u, v \in \mathcal{T}$ according to the generation j of $u_j = u \wedge v \in \mathcal{T}$ the lowest common ancestor of u and v :

$$\mathbb{E}[Z_k^2] = \mathbb{E}\left[\sum_{\substack{u>z\tilde{z}, v>\tilde{z}\\ |u|=|v|=\mathrm{e}^{\lambda(k+1)}}} \mathbb{1}_{\left\{\forall \mathrm{e}^{\lambda k}<i\leq \mathrm{e}^{\lambda(k+1)}, V(u_i)\in I_i, V(v_i)\in I_i\right\}}\right] = \sum_{j=0}^{\ell_k} B_{k,j}, \tag{4.4}$$

where $B_{k,k} = Z_k$ (for each time $v = u = u_j$) and for $j < k$,

$$B_{k,j} := \mathbb{E}\left[\sum_{u>z\tilde{z}, |u|=\mathrm{e}^{\lambda(k+1)}} \mathbb{1}_{\left\{\forall \mathrm{e}^{\lambda k}<i\leq \mathrm{e}^{\lambda(k+1)}, V(u_i)\in I_i\right\}} \sum_{\substack{v>u_j, |v|=\mathrm{e}^{\lambda(k+1)}\\ v_{j+1}\neq u_{j+1}}} \mathbb{1}_{\left\{\forall \mathrm{e}^{\lambda k}+j<i\leq \mathrm{e}^{\lambda(k+1)}, V(v_i)\in I_i\right\}}\right]. \tag{4.5}$$

Thanks to Lemma 2.1, we have :

$$\begin{aligned} h_{k,j}(x) &:= \mathbb{E}\left[\sum_{v\geq u_j, |v|=\mathrm{e}^{\lambda(k+1)}} \mathbb{1}_{\left\{\forall \mathrm{e}^{\lambda k}+j<i\leq \mathrm{e}^{\lambda(k+1)}, V(v_i)\in I_i\right\}} \middle| V(u_j) = x\right] \\ &= \mathbb{E}\left[\mathrm{e}^{S_{\ell_k - j}} \mathbb{1}_{\left\{\forall 0<i\leq \ell_k - j, x+S_i\in I_{\mathrm{e}^{\lambda k}+j+i}\right\}}\right] \\ &\leq \exp\left(a\left(\mathrm{e}^{\lambda(k+1)/3} - (\mathrm{e}^{\lambda k}+j)^{1/3}\right) + b_{\mathrm{e}^{\lambda k}+j}\right) \mathbb{P}\left(\forall 0 < i \leq \ell_k - j, x + S_i \in I_{\mathrm{e}^{\lambda k}+j+i}\right). \end{aligned} \tag{4.6}$$

Actually we need u in order to define $h_{k,j}(x)$ but the real number obtained actually does not depend on the choice of u.

By conditioning on the σ-algebra generated by the ξ_v, $v \in \Gamma(u_i)$, $\mathrm{e}^{\lambda k} \leq i \leq \mathrm{e}^{\lambda k} + j - 1$ for each u, equation (4.5) gives, in the case of deterministic branching :

$$B_{k,j} \leq \sup_{x\in I_{\mathrm{e}^{\lambda k}+j}} h_{k,j}(x)\mathbb{E}\left[Z_k\right].$$

In the general case, this argument fails because the number (and the displacements) of the sisters of u_{j+1} are correlated with $\xi_{u_{j+1}}$ (and with the fact that this individual exists). We have independence of the σ-algebra mentioned above for the descendants of the sisters of u_{j+1}. If we assume that each individual has almost surely at most r children, we obtain

$$\sum_{\substack{v>u_j, |v|=\mathrm{e}^{\lambda(k+1)}\\ v_{j+1}\neq u_{j+1}}} \mathbb{1}_{\left\{\forall \mathrm{e}^{\lambda k}+j<i\leq \mathrm{e}^{\lambda(k+1)}, V(v_i)\in I_i\right\}} \leq (r-1) \sup_{x\in I_{\mathrm{e}^{\lambda k}+j+1}} h_{k,j+1}(x).$$

Hence

$$B_{k,j} \leq (r-1) \sup_{x \in I_{e^{\lambda k}+j+1}} h_{k,j+1}(x)\mathbb{E}\left[Z_k\right].$$

In the case of an unbounded number of children, we remove all the descendants of the individuals having a number of children greater than some number r_k to be set later. This obviously gives a lower bound, and that is what we want. Formally, we keep the same notations and add a superscript $^{(k)}$ when dealing with this new process. Equation 4.4 becomes

$$\mathbb{E}[Z_k^{(k)2}] = \sum_{j=0}^{\ell_k} B_{k,j}^{(k)}, \tag{4.7}$$

and we have the upper bound

$$B_{k,j}^{(k)} \leq (r_k - 1) \sup_{x \in I_{e^{\lambda k}+j+1}} h_{k,j+1}(x)^{(k)}\mathbb{E}\left[Z_k^{(k)}\right],$$

with, obviously from the definition, $h_{k,j}(x)^{(k)} \leq h_{k,j}(x)$.

We define B and C by $B(\alpha, \ell) := \lfloor \alpha\ell \rfloor + 1$ and $C(\alpha, \ell) := \ell$ and write, for any $\alpha \in (0,1)$ $j := B(\alpha, \ell_k) - 1$. Proposition 2.9 (combined with Remark 2.12) yields that, uniformly in $\alpha \in (0,1)$ and $x \in I_{e^{\lambda k}+B(\alpha,\ell_k)}$,

$$\limsup_{k\to\infty} \frac{1}{\ell_k^{1/3}} \mathbb{P}\left(\forall 0 < i \leq \ell_k - (j+1), x + S_i \in I_{e^{\lambda k}+j+1+i}\right) \leq -C_{g_1,g_2}^{\alpha,1}.$$

Combining with the bound (4.6) yields that, uniformly in $\alpha \in (0,1)$,

$$\limsup_{k\to\infty} \frac{1}{\ell_k^{1/3}} \log \frac{B_{k,B(\alpha,\ell_k)-1}^{(k)}}{\mathbb{E}\left[Z_k^{(k)}\right]} \leq g_2(1) - g_2(\alpha) + g(\alpha) - C_{g_1,g_2}^{\alpha,1}. \tag{4.8}$$

4.3 Lower bound for the first moment

For any $k \geq 1$, we consider i.i.d. random variables $X_i^{(k)}$, $1 \leq i \leq \ell_k$ with the same distribution as X conditioned on $\nu \leq r_k$ (with (X, ν) defined in Lemma 2.2) and write $S_j^{(k)} := \sum_{i=1}^{j} X_i^{(k)}$ for any $0 \leq j \leq \ell_k$. Let $\varepsilon > 0$.

By Lemma 2.2,

$$
\begin{aligned}
\mathbb{E}[Z_k^{(k)}] &= \mathbb{E}\left[\sum_{u>\tilde{z},|u|=\mathrm{e}^{\lambda(k+1)}} \mathbb{1}_{\left\{\forall \mathrm{e}^{\lambda k}<i\leq \mathrm{e}^{\lambda(k+1)}, V(u_i)\in I_i, \#\Gamma(u_{i-1})\leq r_k\right\}}\right] \\
&= \mathbb{E}\left[\mathrm{e}^{S_{\ell_k}} \mathbb{1}_{\left\{\forall i\leq \ell_k, a\mathrm{e}^{\lambda k/3}+S_i\in I_{\mathrm{e}^{\lambda k}+i}, \nu_i\leq r_k\right\}}\right] \\
&= \mathbb{P}(\nu\leq r_k)^{\ell_k}\left[\mathrm{e}^{S_{\ell_k}^{(k)}} \mathbb{1}_{\left\{\forall i\leq \ell_k, a\mathrm{e}^{\lambda k/3}+S_i^{(k)}\in I_{\mathrm{e}^{\lambda k}+i}\right\}}\right] \\
&\geq \mathbb{P}(\nu\leq r_k)^{\ell_k}\exp\left(l_k^{1/3}\left(g_2(1)-\varepsilon\right)\right)\mathbb{P}\left(g_1\leq_{\ell_k} s_{\ell_k}^{(k)}\leq_{\ell_k} g_2; S_{\ell_k}^{(k)}\geq l_k^{1/3}\left(g_2(1)-\varepsilon\right)\right),
\end{aligned} \tag{4.9}
$$

where, for any $t\in[0,1]$,

$$
s_{\ell_k}^{(k)}(t) := \frac{S_{\lfloor t l_k\rfloor}^{(k)}}{l_k^{1/3}}.
$$

Let δ_1 and δ_2 be like in condition (1.1), and let $\delta_3 := \frac{\delta_1}{1+\delta_1}$. Hölder's inequality yields

$$
\begin{aligned}
\mathbb{P}(\nu>r_k) &= \mathbb{E}\left[\mathbb{1}_{\{\#\mathcal{T}_1>r_k\}}\sum_{|u|=1}\mathrm{e}^{-\xi_u}\right] \\
&= \mathbb{E}\left[\left(\#\mathcal{T}_1^{\delta_3}\mathbb{1}_{\{\#\mathcal{T}_1>r_k\}}\right)\left(\#\mathcal{T}_1^{-\delta_3}\sum_{|u|=1}\mathrm{e}^{-\xi_u}\right)\right] \\
&\leq \mathbb{E}\left[\#\mathcal{T}_1\mathbb{1}_{\{\#\mathcal{T}_1>r_k\}}\right]^{\delta_3}\mathbb{E}\left[\left(\#\mathcal{T}_1^{-\delta_3}\sum_{|u|=1}\mathrm{e}^{-\xi_u}\right)^{1+\delta_1}\right]^{\frac{1}{1+\delta_1}}.
\end{aligned} \tag{4.10}
$$

We begin with the second factor in the right-hand side of (4.10). The convexity of $t\mapsto t^{1+\delta_1}$ gives

$$
\#\mathcal{T}_1^{-\delta_1}\left(\sum_{|u|=1}\mathrm{e}^{-\xi_u}\right)^{1+\delta_1}\leq\sum_{|u|=1}\mathrm{e}^{-\xi_u(1+\delta_1)}.
$$

Hence

$$
\mathbb{E}\left[\left(\#\mathcal{T}_1^{-\delta_3}\sum_{|u|=1}\mathrm{e}^{-\xi_u}\right)^{1+\delta_1}\right]\leq\Phi(1+\delta_1)<+\infty.
$$

For the first factor in the right-hand side of (4.10), Markov's inequality yields

$$
\mathbb{E}\left[\#\mathcal{T}_1\mathbb{1}_{\{\#\mathcal{T}_1>r_k\}}\right]\leq\frac{\mathbb{E}\left[\#\mathcal{T}_1^{1+\delta_2}\right]}{r_k^{\delta_2}}.
$$

Finally, the bound (4.10) becomes

$$\mathbb{P}(\nu > r_k) \leq \frac{\mathbb{E}\left[\#\mathcal{T}_1^{1+\delta_2}\right]^{\delta_3}}{r_k^{\delta_2\delta_3}} \Phi(1+\delta_1)^{\frac{1}{1+\delta_1}}.$$

We choose $r_k := \lfloor e^{l_k^{1/4}} \rfloor$. Therefore

$$\lim_{k\to\infty} \frac{1}{\ell_k^{1/3}} \log \mathbb{P}(\nu \leq r_k)^{l_k} = 0.$$

In order to apply Proposition 2.16 to the third factor of (4.9), we have to check the conditions (2.7) of Lemma 2.14. Notice that for $t \in [-1, \delta_2]$, $\mathbb{E}\left[e^{tX}\right] = \Phi(1+t) < +\infty$ and

$$\mathbb{E}\left[e^{tX_1^{(k)}}\right] = \frac{\mathbb{E}\left[\sum e^{(1+t)\xi} \mathbb{1}_{\{\#\mathcal{T}_1 > r_k\}}\right]}{\mathbb{P}(\nu \leq r_k)} \leq \frac{\Phi(1+t)}{\mathbb{P}(\nu \leq r_k)}.$$

This implies that $\sup_{k\geq 1} \mathbb{E}\left[\left|X_1^{(k)}\right|^3\right] < +\infty$.

Since X is centered,

$$\mathbb{E}\left[X_1^{(k)}\right] = \frac{\mathbb{E}\left[X \mathbb{1}_{\{\nu \leq r_k\}}\right]}{\mathbb{P}(\nu \leq r_k)} = -\frac{\mathbb{E}\left[X \mathbb{1}_{\{\nu > r_k\}}\right]}{\mathbb{P}(\nu \leq r_k)}.$$

Recall that X admits exponential moments, hence moments of all orders. Cauchy-Schwarz inequality yields

$$\mathbb{E}\left[X \mathbb{1}_{\{\nu > r_k\}}\right] \leq \sqrt{\mathbb{E}\left[X^2\right] \mathbb{P}(\nu > r_k)} = O\left(r_k^{-\delta_2\delta_3/2}\right) = o\left(\ell_k^{-2/3}\right).$$

This inequality, combined with the previous one, proves the second condition in (2.7). It remains to check the last one, which is a consequence of $\mathbb{E}\left[X_1^{(k)}\right] \to 0$ and $\mathbb{E}\left[X_1^{(k)2}\right] \to \mathrm{Var}(X) = \sigma^2$.

With the notations of Lemma 2.15

$$\begin{aligned} & \mathbb{P}\left(g_1 \leq_{\ell_k} s_{\ell_k}^{(k)} \leq_{\ell_k} g_2; S_{\ell_k}^{(k)} \geq \ell_k^{1/3}\left(g_2(1) - \varepsilon\right)\right) \\ \geq\ & P_{\ell_k}(M, \varepsilon_1, \varepsilon_2) \inf_{-M\varepsilon_2\ell_k^{1/3} \leq z \leq -\varepsilon_1\varepsilon_2\ell_k^{1/3}} Q_{\ell_k}(M, \varepsilon_1, \varepsilon_2, g_1, g_2) \end{aligned}$$

where

$$\begin{aligned} Q_{\ell_k}(M, \varepsilon_1, \varepsilon_2, g_1, g_2) := \mathbb{P}^z \Bigg(& S_{\ell_k - \lfloor \varepsilon_2 \ell_k^{1/3} \rfloor}^{(k)} \geq \ell_k^{1/3}\left(g_2(1) - \varepsilon\right); \\ & \forall i \leq \ell_k - \lfloor \varepsilon_2 \ell_k^{1/3} \rfloor, g_1\left(\frac{\lfloor \varepsilon_2 \ell_k^{1/3} \rfloor + i}{\ell_k}\right) \leq \frac{S_i^{(k)}}{\ell_k^{1/3}} \leq g_2\left(\frac{\lfloor \varepsilon_2 \ell_k^{1/3} \rfloor + i}{\ell_k}\right)\Bigg). \end{aligned}$$

We estimate $Q_{\ell_k}(M,\varepsilon_1,\varepsilon_2,g_1,g_2)$ by Proposition 2.16 with $\beta = 0$, $\gamma = 1$, $B(\ell_k) = \lfloor \varepsilon_2 \ell_k^{1/3} \rfloor$, $C(\ell_k) = \ell_k$, $u_{\ell_k} = -M\varepsilon_2 \ell_k^{1/3}$ and $v_{\ell_k} = -\varepsilon_1\varepsilon_2\ell_k^{1/3}$. Then letting $\varepsilon_2 \to 0$ yields

$$\liminf_{k\to\infty} \frac{1}{\ell_k^{1/3}} \log \mathbb{E}\left[Z_k^{(k)}\right] \geq g_2(1) - \varepsilon - C_{g_1,g_2}.$$

This inequality holds for any $\varepsilon > 0$ small enough, hence also for $\varepsilon = 0$.

4.4 Proof of Proposition 1.4

Combining with (4.8) yields that, uniformly in $\alpha \in (0,1)$,

$$\limsup_{k\to\infty} \frac{1}{\ell_k^{1/3}} \log \frac{B^{(k)}_{k,B(\alpha,\ell_k)-1}}{\left(\mathbb{E}\left[Z_k^{(k)}\right]\right)^2} \leq -g_2(\alpha) + g(\alpha) + C^{0,\alpha}_{g_1,g_2}.$$

Consequently, in view of (4.7) and (4.2)

$$\limsup_{k\to\infty} \frac{1}{\ell_k^{1/3}} \log A_k^{(k)} \geq \min_{0\leq\alpha\leq 1} g_2(\alpha) - g(\alpha) - C^{0,\alpha}_{g_1,g_2}.$$

Lemma 4.1 yields

$$\max_{0\leq\alpha\leq 1} G_\lambda(\alpha) < 0 \Rightarrow \mathbb{P}\left(\exists u \in \mathcal{T}_\infty, \forall i \geq 1, V(u_i) \leq a i^{1/3}\right) > 0 \tag{4.11}$$

where

$$G_\lambda(\alpha) := -g_2(\alpha) + g(\alpha) + \frac{\pi^2\sigma^2}{2}\int_0^\alpha \frac{\mathrm{d}t}{g(t)^2} + \mathrm{e}^{-\lambda/3}\left[-g_2(1) + \frac{\pi^2\sigma^2}{2}\int_0^1 \frac{\mathrm{d}t}{g(t)^2}\right].$$

We denote

$$\forall t \in [0,1], f(t) := \left(t + \frac{1}{\mathrm{e}^\lambda - 1}\right)^{1/3}.$$

We have $g_2 = af - af(0)$. We choosed for the width of the pipe the function $g := bf$. This gives

$$G_\lambda(\alpha) = af(0) + (b-a)f(\alpha) + \frac{\pi^2\sigma^2}{2b^2}\int_0^\alpha \frac{\mathrm{d}t}{f(t)^2} + \mathrm{e}^{-\lambda/3}\left[af(0) - af(1) + \frac{\pi^2\sigma^2}{2b^2}\int_0^1 \frac{\mathrm{d}t}{f(t)^2}\right].$$

Since $f(1) = \mathrm{e}^{\lambda/3} f(0)$ and $f' = \frac{1}{3}f^{-2}$, this becomes :

$$G_\lambda(\alpha) = \left(b + \frac{3\pi^2\sigma^2}{2b^2} - a\right) f(\alpha) + \mathrm{e}^{-\lambda/3}\left[af(0) - \frac{3\pi^2\sigma^2}{2b^2} f(0)\right].$$

Assuming $a > a_c$, we can choose b such that $b + \frac{3\pi^2\sigma^2}{2b^2} < a$. Since f is increasing on $[0,1]$,

$$\max_{0\le\alpha\le1} G_\lambda(\alpha) = G_\lambda(0) = f(0)\left[\left(b + \frac{3\pi^2\sigma^2}{2b^2} - a\right) + \mathrm{e}^{-\lambda/3}\left(a - \frac{3\pi^2\sigma^2}{2b^2}\right)\right].$$

This value is negative for sufficiently large λ (that we can choose such that we also have $\mathrm{e}^\lambda \in \mathbb{N}$), which, in view of (4.11), completes the proof.

5 The extinction rate

Throughout this section, we assume $a < a_c$.

5.1 Upper bound

It follows from the computations of Section 3 that, for any continuous function $g : [0,1] \mapsto [0,+\infty)$ such that $g(0) = 1$,

$$\limsup_n \frac{1}{n^{1/3}} \log \mathbb{P}\left(\exists u \in \mathcal{T}_n, \forall i \le n, V(u_i) \le a i^{1/3}\right) \le -c_g,$$

where

$$c_g := \min_{0\le t\le1}\left(g(t) + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{\mathrm{d}u}{g(u)^2} - a t^{1/3}\right).$$

The best choice for g is the one described in the end of Section 3 : it is the solution of the integral equation (3.11) with $s = c_g$ such that $g(1) = 0$ (or equivalently, $t_{\max} = 1$). We can make this choice thanks to Proposition 3.2 and Proposition 3.6(iii).

5.2 Lower bound

For the sake of clarity, we treat only the regular case. The modifications required by the general case are the same as above. We directly apply the Paley-Zygmund inequality to the number W_n of individuals $u \in \mathcal{T}_n$ such that

$$\forall i \le n, a i^{1/3} - n^{1/3} g(\frac{i}{n}) \ge V(u) \le a i^{1/3}.$$

Following the computations of Section 4, we obtain

$$\liminf_n \frac{1}{n^{1/3}} \log \mathbb{P}\left(\exists u \in \mathcal{T}_n, \forall i \le n, V(u_i) \le a i^{1/3}\right) \ge \liminf_n \frac{1}{n^{1/3}} \log\left(\mathbb{P}(W_n \ge 1) \ge\right) \ge -d_g,$$

where

$$d_g := \max_{0\le t\le1}\left(g(t) + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{\mathrm{d}u}{g(u)^2} - a t^{1/3}\right).$$

The optimal g would be exactly the same as in the upper bound, except that we are forced to take approximations because g must be strictly positive on $[0,1]$. Since this optimal g is such that $g(t)+\frac{\pi^2\sigma^2}{2}\int_0^t\frac{\mathrm{d}u}{g(u)^2}-at^{1/3}$ does not depend on t, we have proved

$$c:=\sup_g c_g=\inf_g d_g.$$

This completes the proof of Proposition 1.5.

6 Some refinements

6.1 About more general barriers

We are going to give a sketch of the proof of Theorem 1.8. The main idea is to consider the function g_2 defined by

$$\forall t\in[0,1), g_2(t):=a^+t^{1/3};\quad g_2(1)=a^-.$$

We compute the quantities R_j and follow the arguments of Section 3. Almost everything goes as before with $a=a^+$, except that R_∞ is less than before. We search for the optimal g (with still $g_1=g_2-g$). This is a solution of 3.11 (with a^+ instead of a) over $[0,1]$, but the boundary condition $g(1)=0$ is replaced by $g(1-)=a^+-a^-$. If a^- is the critical value $\frac{3\pi^2\sigma^2}{2b_{a^+}}$ of Theorem 1.8, then the function $g(t)=b_{a^+}t^{1/3}$ almost works, but the factor giving the exponential decay of the probability is $0=g(0)$. If a^- is smaller than the critical value, we obtain some solution (starting with the boundary condition at 1 and solving the differential equation) with $g(0)>0$, which implies that there is extinction and that, roughly speaking, the probability decays at least like $\exp\Big(-g(0)n_k^{1/3}(1+o(1))\Big)$ along a subsequence n_k such that $\varphi(n_k)/n_k^{1/3}$ is close to a^-.

Conversely, when we consider the same function g_2 with a^- greater than the critical value, we can find a solution of the differential equation with arbitrarily small $g(0)>0$ such that $g(1-)>a^+-a^-$, which means that the probability to have an exponential population $\exp\Big((a^--a^++g(1-))n_k^{1/3}(1+o(1))\Big)$ is of order $\exp\Big(-g(0)n_k^{1/3}(1+o(1))\Big)$. Then we can apply the arguments of Section 4.

6.2 Sketch of the proof of Proposition 1.6

First we give the upper bound. Let $a<a_c$ and let g be with the optimal function seen before. Let $n\geq 1$. We add a second absorbing barrier $i\mapsto ai^{1/3}-n^{1/3}g(\frac{i}{n})$. We write Z_i [resp.

Z_i^*] for the number of individuals in generation i that survive below the barrier $i \mapsto ai^{1/3}$ [resp. between the barriers]. We have

$$\mathbb{P}(Z > k) \leq \mathbb{P}(Z_{n+1} > 0) + \sum_{i=0}^{n} \mathbb{P}\left(Z_i^* > \frac{k}{n+1}\right) + \mathbb{P}(Z_i > Z_i^*).$$

The terms $\mathbb{P}(Z_i > Z_i^*)$ correspond to the R_j. We know that they are, like $\mathbb{P}(Z_{n+1} > 0)$, $\exp\left(-g(0)n^{1/3}(1+o(1))\right)$. It is not hard to see from the integral equation satisfied by g that

$$\mathbb{E}\left[Z^*_{\lfloor \alpha n \rfloor}\right] = \exp\left((g(\alpha) - g(0))n^{1/3}(1+o(1))\right).$$

Therefore, by Markov's inequality,

$$\sum_{i=0}^{n} \mathbb{P}(Z_i^* > \frac{k}{n+1}) \leq \frac{(n+1)^2}{k} \exp\left((d-c)n^{1/3}(1+o(1))\right).$$

Finally we choose a sequence $n = n(k)$ such that $k \sim \exp\left(dn^{1/3}\right)$, and we obtain the upper bound by letting $k \to \infty$.

For the lower bound, we consider the same barriers as above. For any $\alpha \in [0,1]$, the probability that at least one individual survives between the barriers until generation $\lfloor \alpha n \rfloor$ and is close to the lower barrier at time $\lfloor \alpha n \rfloor$ is $\exp\left(cn^{1/3}(1+o(1))\right)$. We choose α maximizing g. An individual close to the lower barrier at time $\lfloor \alpha n \rfloor$ gives, with probability at least $\exp\left((c-\varepsilon_1)n^{1/3}\right)$, in around $\varepsilon_2 n^{1/3}$ generations a number of children at least $\exp\left((d-\varepsilon_3)n^{1/3}\right)$. Taking ε_1, ε_2, ε_3 small and making the same choice of n as for the upper bound yield the result.

6.3 Proof of Corollary 1.3 from Theorem 1.2

Lemma 6.1. *We assume that the underlying Galton-Watson process is supercritical, and we denote by q the extinction probability. Then*

$$\forall a \in \mathbb{R}, \mathbb{P}\left(\inf_{u \in \mathcal{T}_\infty} \limsup_{n \to \infty} \frac{V(u_n)}{n^{1/3}} > a\right) \in \{q, 1\}.$$

In particular, when there is extinction, $\mathcal{T}_\infty = \emptyset$ and we follow the usual convention which says that the infimum of the empty set is $+\infty$.

Proof of Lemma 6.1. Let f be the generating function of the underlying supercritical Galton-Watson process. We mean that for any $s \in [0,1]$, $f(s) = \mathbb{E}\left[s^{\#\mathcal{T}_1}\right]$. It is well known that f

has exactly two fixed points, 1 and $q \in [0,1)$ the extinction probability. Let $a \in \mathbb{R}$. In order to prove the lemma, it suffices to show that the number

$$\mathbb{P}\left(\inf_{u\in\mathcal{T}_\infty}\limsup_{n\to\infty}\frac{V(u_n)}{n^{1/3}} > a\right)$$

is a fixed point of the generating function f.

We write the boundary of the tree

$$\mathcal{T}_\infty = \bigcup_{|v|=1}\mathcal{T}_\infty^v,$$

where

$$\mathcal{T}_\infty^v = \{u \in \mathcal{T}_\infty, u > v\}$$

is the boundary of the tree $\mathcal{T}^v$ rooted at v. Hence

$$\inf_{u\in\mathcal{T}_\infty}\limsup_{n\to\infty}\frac{V(u_n)}{n^{1/3}} = \inf_{|v|=1}\inf_{u\in\mathcal{T}_\infty^v}\limsup_{n\to\infty}\frac{V(u_n)}{n^{1/3}} = \inf_{|v|=1}\inf_{u\in\mathcal{T}_\infty^v}\limsup_{n\to\infty}\frac{V(v)+V^v(u_n)}{n^{1/3}}. \tag{6.1}$$

For any $v \in \mathcal{T}_1$, $\frac{V(v)}{n^{1/3}} \to 0$, hence for any $u \in \mathcal{T}_\infty^v$,

$$\limsup_{n\to\infty}\frac{V(v)+V^v(u_n)}{n^{1/3}} = \limsup_{n\to\infty}\frac{V^v(u_n)}{(n-1)^{1/3}}.$$

From this last equality and the independence properties of the branching random walk, we deduce that the random variables

$$\inf_{u\in\mathcal{T}_\infty^v}\limsup_{n\to\infty}\frac{V(u_n)}{n^{1/3}}, v \in \mathcal{T}_1$$

form an i.i.d. family, and are distributed like $\inf_{u\in\mathcal{T}_\infty}\limsup_{n\to\infty}\frac{V(u_n)}{n^{1/3}}$.

With this in mind, equation (6.1) yields, for any $a \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}\left(\inf_{u\in\mathcal{T}_\infty}\limsup_{n\to\infty}\frac{V(u_n)}{n^{1/3}} > a\right) &= \mathbb{P}\left(\forall v \in \mathcal{T}_1 \inf_{u\in\mathcal{T}_\infty^v}\limsup_{n\to\infty}\frac{V(u_n)}{n^{1/3}} > a\right)\\
&= \mathbb{E}\left[\prod_{v\in\mathcal{T}_1}\mathbb{P}\left(\inf_{u\in\mathcal{T}_\infty^v}\limsup_{n\to\infty}\frac{V(u_n)}{n^{1/3}} > a\right)\right]\\
&= \mathbb{E}\left[\mathbb{P}\left(\inf_{u\in\mathcal{T}_\infty}\limsup_{n\to\infty}\frac{V(u_n)}{n^{1/3}}\right)^{\#\mathcal{T}_1}\right]\\
&= f\left(\mathbb{P}\left(\inf_{u\in\mathcal{T}_\infty}\limsup_{n\to\infty}\frac{V(u_n)}{n^{1/3}}\right)\right).
\end{aligned}$$

This completes the proof of the lemma. □

Proof of Corollary 1.3. Let $a > a_c$. By Theorem 1.2, the branching random walk absorbed by the barrier $i \mapsto ai^{1/3}$ survives with positive probability. In other words, with positive probability,

$$\exists u \in \mathcal{T}_\infty \forall n \geq 1, \frac{V(u_n)}{n^{1/3}} \leq a.$$

Hence, with at least the same positive probability,

$$\exists u \in \mathcal{T}_\infty \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}} \leq a.$$

In light of the lemma, this implies that

$$\mathbb{P}\left(\inf_{u\in\mathcal{T}_\infty} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}} > a\right) = q$$

or, equivalently, that on the set of ultimate survival of the underlying Galton-Watson tree, the random variable

$$\inf_{u\in\mathcal{T}_\infty} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}}$$

is almost surely less than or equal to a, for any $a > a_c$, hence also for $a = a_c$.

It remains to prove that this random variable is at least a_c, almost surely (remind that it equals $+\infty$ when there is extinction). We reason ab absurdo. We assume that with positive probability, this random variable is less than some positive real $a < a_c$. We deduce the existence of an integer N such that, with positive probability,

$$\exists u \in \mathcal{T}_\infty \forall n \geq N, \frac{V(u_n)}{n^{1/3}} \leq a.$$

Focusing on the value of $V(u_n)$ on this event yields that there exist some real x satisfying the following conditions :

(i) With positive probability, there exists $u \in \mathcal{T}_N$ such that $V(u) \leq x$;

(ii) With positive probability, there exists $v \in \mathcal{T}_\infty$ such that $\forall i \geq 1, V(v_i)+x \leq a(N+i)^{1/3}$.

If condition (ii) holds for some x, then it obviously also holds for any smaller value. Hence, since (i) holds for $x = 0$, we may assume $x \leq 0$ (if $x > 0$ we take $x = 0$).

Actually condition (i) is equivalent to $\mathbb{P}\left(\exists u \in \mathcal{T}_1 : V(u) \leq \frac{x}{N}\right) > 0$ and implies that with positive probability

$$\exists u \in \mathcal{T}_N \forall n \leq N, \xi_{u_n} \leq \frac{x}{N}.$$

Hence, with at least the same probability,

$$\exists u \in \mathcal{T}_N \forall n \leq N, V(u_n) \leq \frac{xn}{N} \leq 0 \leq an^{1/3}.$$

This, combined with (ii) yields that the branching random walk absorbed by the barrier $i \mapsto ai^{1/3}$ survives with positive probability.

By Theorem 1.2, this implies $a \leq a_c$, which contradicts our assumption $a < a_c$. □

7 Extension of the results to the non critical case

We assume $\Phi(0) < +\infty$ and $\zeta := \sup\{t : \Phi(t) < +\infty\} > 0$. We define, for $0 < t < \zeta$,

$$F(t) := \frac{\Psi(t)}{t}$$

Lemma 7.1. *If equation (1.2) does not hold for any t^*, then $\forall 0 < t < \zeta, t\Psi'(t) < \Psi(t)$ and F is decreasing and convex.*

Proof. For $t > 0$ very small, $t\Psi'(t) - \Psi(t) < 0$, since by convexity of Ψ, either $\Psi'(0) \in \mathbb{R}$ or $\Psi'(0) = -\infty$. $\forall t > 0$, $t\Psi'(t) - \Psi(t) \neq 0$ by hypothesis, and so by continuity, it is always negative. As a consequence, for any $0 < t < \zeta$,

$$\begin{aligned} F'(t) &= \frac{t\Psi'(t) - \Psi(t)}{t^2} < 0; \\ F''(t) &= \frac{t^2\Psi''(t) + 2(\Psi(t) - t\Psi'(t))}{t^3} > 0; \end{aligned}$$

which ends the proof of the lemma. □

We are now ready to determine whether a branching random walk can be applied the reduction of Section 1.2. It easy to construct examples for any of the cases studied below.

7.1 Case $\zeta < +\infty$

a) If $\zeta < +\infty$ and $\Phi(\zeta) = +\infty$, then, by Fatou's lemma, $\lim_{t\to\zeta} \Phi(t) = +\infty$.

b) If $\zeta < +\infty$, $\Phi(\zeta) < +\infty$ and $\Phi'(\zeta) = +\infty$.

In these two cases, it is easy to deduce from the lemma that we can find some $t^* \in (0, \zeta)$ such that equation (1.2) holds.

When $\zeta < +\infty$, $\Phi(\zeta) < +\infty$ and $\Phi'(\zeta) < +\infty$, it depends on the sign of $\zeta\Psi'(\zeta) - \Psi(\zeta)$:

c) If $\zeta\Psi'(\zeta) - \Psi(\zeta) > 0$, then by continuity we can find some $t^* \in (0, \zeta)$ such that equation (1.2) holds.

d) If $\zeta\Psi'(\zeta) - \Psi(\zeta) < 0$, then it is impossible to find such a t^* (because $t \mapsto t\Psi'(t) - \Psi(t)$ is increasing) and consequently the reduction to the critical case does not work.

e) If $\zeta\Psi'(\zeta) - \Psi(\zeta) = 0$, then $t^* = \zeta$ works, but we still have to check whether $\widetilde{\sigma}^2$ is finite or not.

It is easy to construct examples fitting any of these five cases.

7.2 Case $\zeta = +\infty$

In this case, we can be much more precise and tell whether we can find some $t^* \in (0, \zeta)$ such that equation (1.2) holds directly from the intensity measure μ of the point process $\{\xi_u, |u| = 1\}$

Define $x_{\min} = \inf\{x \in \mathbb{R}, \mu((-\infty, x)) > 0\}$ the minimum of the support of μ (and $-\infty$ if μ is not lower bounded). It is clear that $\lim_{t\to+\infty} F(t) = -x_{\min}$. If $x_{\min} > -\infty$ we will consider $\mu(\{x_{\min}\})$ the mass of the eventual atom of μ in $x_{\min}$. We can now state :

Proposition 7.2. *There is some $t^* \in (0, \zeta)$ such that equation (1.2) holds if and only if $x_{\min} > -\infty$ or $\mu(\{x_{\min}\}) < 1$.*

Proof. We distinguish four cases :

a) If $x_{\min} = -\infty$, then $\lim_{+\infty} F = -x_{\min} = +\infty$. Consequently F can not be decreasing.

b) If $x_{\min} > -\infty$ and $\mu(\{x_{\min}\}) = 0$. We still have $\lim_{+\infty} F = -x_{\min}$. Almost surely, for all $u \in \mathcal{T}_1$, $\xi_u < x_{\min}$, hence $\sum_{|u|=1} \mathrm{e}^{t(x_{\min}-\xi_u)} \to 0$ as $t \to +\infty$. By the monotone convergence theorem $\lim_{t\to+\infty} \mathbb{E}\left[\sum_{|u|=1} \mathrm{e}^{t(x_{\min}-\xi_u)}\right] = 0$. This implies that for t large enough,

$$F(t) = -x_{\min} + \frac{1}{t}\log\left(\mathbb{E}\left[\sum_{|u|=1} \mathrm{e}^{t(x_{\min}-\xi_u)}\right]\right) < -x_{\min}.$$

c) If $x_{\min} > -\infty$ and $0 < \mu(\{x_{\min}\}) < 1$, we can write

$$F(t) = -x_{\min} + \frac{1}{t}\log\left(\mu(\{x_{\min}\}) + \varepsilon(t)\right)$$

where $\varepsilon(t) := \mathbb{E}\left[\sum_{|u|=1} \mathbb{1}_{\xi_u > x_{\min}} \mathrm{e}^{t(x_{\min}-\xi_u)}\right]$ decreases to 0 as t increases to infinity. Like in case b), for t large enough, $\log\left(\mu(\{x_{\min}\}) + \varepsilon(t)\right) < 0$ and $F(t) < -x_{\min}$.

In these three cases, we have proved that F is not decreasing, and we conclude thanks to Lemma 7.1 : there is some $t^* \in (0, \zeta)$ such that equation (1.2) hold, since otherwise the lemma would imply that F is decreasing, which is false.

d) When $x_{\min} > -\infty$ and $\mu(\{x_{\min}\}) \geq 1$, we still have

$$\Psi(t) = -tx_{\min} + \log\left(\mu(\{x_{\min}\}) + \varepsilon(t)\right) \geq -x_{\min},$$

where $\varepsilon(t) := \mathbb{E}\left[\sum_{|u|=1} \mathbb{1}_{\xi_u > x_{\min}} \mathrm{e}^{t(x_{\min}-\xi_u)}\right]$ decreases to 0 as t increases to infinity. By convexity, Ψ' increases, and so converges to $-x_{\min}$ as t goes to $+\infty$.

Finally, for any $t > 0$, we proved that $\Psi'(t) < -x_{\min}$ whereas $\Psi(t) \geq -tx_{\min}$, hence $t\Psi'(t) < \Psi(t)$. □

Remark 7.3. When $x_{\min} > -\infty$ and $\mu(\{x_{\min}\}) > 1$, we obtain, by letting only individuals with minimal displacement reproduce and killing the others, a supercritical Galton-Watson subtree. Hence with positive probability, there is an infinite path through only such individuals. Besides, conditional on survival, there exists almost surely an individual from which we can start an infinite path down the tree through individuals with minimal displacement. In this case, it is easy, given a barrier, to determine whether the absorbed process survives or not.

When $x_{\min} > -\infty$ and $\mu(\{x_{\min}\}) = 1$, the Galton-Watson subtree described just above is critical. The behaviour is obviously not exactly the same, but is very close. Under the additional assumptions that the displacements are bounded and lattice and that the number of children has finite variance, Theorem 9 from Dekking and Host [27] yields that, almost surely,

$$\min_{|u|=n} V(u) = nx_{\min} + \frac{C}{\log 2} \log\log n + o(1),$$

where $C := \inf\{x > 0 : \mu(x_{\min} + x) > 0\}$. Therefore, in this case, the second order critical barrier is of the order of $\log\log n$ instead of $n^{1/3}$. In the non lattice case, C may be 0 and when this happens, the behaviour is expected to be even closer to the one of the case $\mu(\{x_{\min}\}) > 1$.

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Chapter 4

The upper tail of the normalized minimal displacement

The upper tail of the normalized minimal displacement of branching random walks

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Summary. We study a branching random walk on $\mathbb{R}$ with null speed. We are interested in the minimum over the individuals u in generation n of $\overline{V}(u) := \max_{v \le u} V(v)$ where $V(v)$ denotes the position of the individual v. Fang and Zeitouni [39] proved that this minimum displacement, normalized by $n^{1/3}$, converges almost surely to a constant b_c and establish a small deviation result. We are interested in moderate deviations. We prove that the rate of decay of the probability of having a branching random walk faster than the typical behavior depends heavily on the upper tail of the law of the displacements, being for example exponential in $n^{1/3}$ when the displacements have an exponential upper tail but giving a double exponential when displacements are bounded from above.

Key words. Branching random walk, moderate deviations.

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1 Introduction

We consider a unidimensional branching random walk. The population forms a Galton-Watson tree $\mathcal{T}$, with the probability of a given indivicual to have exactly i children being p_i. For any $u \in \mathcal{T}$ we write $|u|$ for the generation of u. Additionally, for any $i \in \mathbb{N}$ such that $0 \le i \le |u|$, we denote by u_i the ancestor of u in generation i. We denote by $\mathcal{T}_n := \{u \in \mathcal{T} : |u| = n\}$ the individuals in generation n.

We assume that the Galton-Watson tree is supercritical, that is that

$$\sum_{i=1}^{\infty} i p_i > 1.$$

We define $m := \inf\{i \in \mathbb{N}, p_i > 0\}$ the minimum number of children. We assume $m \ge 1$, in other words, the tree has no leaves.

To each individual u in $\mathcal{T}$ except the root, we assign a displacement X_u, where the X_u (with u running over all the potential elements of the random tree $\mathcal{T}$ except de root) are i.i.d. copies of a given real random variable X. We also require that the X_u are independent of the Galton-Watson tree. To be more precise, we can write $i \in \{1,2,\ldots\}$ for the i^{th} child of the root and recursively, if $u = (i_1, i_2, \ldots, i_n) \in \mathcal{T}_n \subset \{1,2,\ldots\}^n$, then we write $(u,i) = (i_1, i_2, \ldots, i_n, i) \in \mathcal{T}_{n+1} \subset \{1,2,\ldots\}^{n+1}$ for the i^{th} child of u. This labelling (called Ulam-Harris labelling or lexicographic labelling) gives a countable tree of potential individuals. Then we define the displacements as an an i.i.d. family of copies of X, indexed by this tree (without the root) and independent of the Galton-Watson tree $\mathcal{T}$.

The root is set at the origin, thus the position of an individual u is the sum of the displacements of u and of the ancestors of u (except the root) :

$$V(u) := \sum_{v \le u} X_v.$$

We are also interested in the maximal position of the random walk along the path from the root to u :

$$\overline{V}(u) := \max_{v \le u} V(v).$$

We define the log-moment generating function function :

$$\Lambda_X(t) := \log \mathbb{E}\left[\mathrm{e}^{tX}\right].$$

We assume that $\Lambda_X(t)$ is finite for some $t_- < -1$ and for some $t_+ > 0$. We consider only the boundary case, which means that we assume the following condition :

$$\Lambda(-1) \sum_{i=1}^{\infty} i p_i = 1 \text{ and } \Lambda'(-1) = 0.$$

Actually this condition is not very restrictive since, for a large class of branching random walks, it holds after an affine modification of X.

We also assume that the law of reproduction admits some moment of order greater that one :

$$\exists \varepsilon > 0, \sum_{i=1}^{\infty} i^{1+\varepsilon} p_i < +\infty.$$

A known result is that, under these assumptions,

$$\frac{\min_{|u|=n} \overline{V}(u)}{n^{1/3}} \to b_c := \left(\frac{3\pi^2\sigma^2}{2}\right)^{1/3} \text{ a.s. as } n \to \infty. \tag{1.1}$$

where

$$0 < \sigma^2 := \sum_{i=1}^{\infty} i p_i \mathbb{E}\left[X^2 \mathrm{e}^{-X}\right] < +\infty. \tag{1.2}$$

Fang and Zeitouni [39] proved this result for regular trees. Under the moment condition on the law of reproduction (and $m \geq 1$ which avoids conditioning on non extinction), this can be extended by using technical arguments developped in [44]. Similar arguments and results can be found in Chapter 3.
In the proof of the result above appears the following estimate :

$$\forall b < b_c, \frac{1}{n^{1/3}} \log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) \leq b n^{1/3}\right) \to b - b_c.$$

Our aim is to estimate the right tail :

$$\mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > b n^{1/3}\right) \tag{1.3}$$

for $b > b_c$.

We expect a faster rate of decay than for the left tail of the distribution of the normalized displacement because, for the minimum to be large, the potential needs to be large along all the paths from the root whereas it suffices to have a small potential along one path to obtain a small minimum.

The answer to our problem actually depends on the right tail of the distribution of X and on m. We obtain a very fast decay, (1.3) being of the order of $\exp(-\exp n^{1/3})$ when X is upperly bounded and $m \geq 2$ (see Theorem 3.1). We still obtain a fast rate of decay, like $\exp\left(-n^{\kappa/3}\right)$ with $\kappa > 1$ when $m \geq 2$ and the right tail of X is of Gaussian-type (see Theorem 4.1). When $m = 1$ or the distribution of X has an exponential right tail, the rate of decay of (1.3) is of the same order $\exp\left(-n^{1/3}\right)$ as for the left tail.

2 Some general bounds

For any $u \in \mathcal{T}$, we denote by $\mathcal{T}^u := \{v \in \mathcal{T} : v \geq u\}$ the subtree of $\mathcal{T}$ constituted by the descendants of u. The relative depth of $v \in \mathcal{T}^u$ is $|v|_u := |v| - |u|$. Accordingly, we define, for any $n \in \mathbb{N}$, $\mathcal{T}_n^u := \{v \in \mathcal{T}^u : |v|_u = n\} = \mathcal{T}^u \bigcap \mathcal{T}_{|u|+n}$, $V^u(v) := \sum_{u<w\leq v} X_w = V(v) - V(u)$ and $\overline{V}^u(v) := \max_{u\leq w \leq v} V^u(w)$.

We write $\widetilde{\mathcal{T}}$ for the subtree of $\mathcal{T}$ obtained by keeping only the first m children of each individual and, for $N \geq 1$, $\widetilde{\mathcal{T}}_N = \widetilde{\mathcal{T}} \bigcap \mathcal{T}_N$ for the corresponding N^{th} generation.

For $x > 0$ and $\beta \in (0,1]$, we define the following event :

$$L_N^\beta(x) := \left\{\# \left\{u \in \mathcal{T}_N : \overline{V}(u) \leq x\right\} < \beta m^N\right\}. \tag{2.1}$$

We use the following proposition in the other sections. It is the spine of the proof whatever the right tail distribution of X. We shall see N as very small before n, either of the order of a constant or tending to ∞ very slowly.

Proposition 2.1. *Let n and N be integers such that $1 \leq N \leq n$.*
Let $\beta \in (0,1]$, b, b_1 and b_2 be real numbers such that $b_1 + b_2 = b$.
The following two inequalities hold :
(2.2)

$$\mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right) \geq \mathbb{P}\left(\min_{|u|=N} \overline{V}(u) > b_1 n^{1/3}, \#\mathcal{T}_N = m^N\right) \mathbb{P}\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right)^{m^N};$$

$$\mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right) \leq \mathbb{P}\left(L_N^\beta(b_1 n^{1/3})\right) + \mathbb{P}\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right)^{\beta m^N}. \tag{2.3}$$

Proof. We notice that for any $u \in \mathcal{T}_n$ and any $0 \leq N \leq |v| = n$, we have

$$\overline{V}(u) = \max\{\overline{V}(u_N), V(u_N) + \overline{V}^{u_N}(u)\} \leq \overline{V}(u_N) + \overline{V}^{u_N}(u). \tag{2.4}$$

As a consequence,

$$\{\overline{V}(u) > bn^{1/3}\} \supset \{\overline{V}(u_N) > b_1 n^{1/3}\} \cap \{\overline{V}^{u_N}(u) > b_2 n^{1/3}\}.$$

Taking the intersection over the $u \in \mathcal{T}_n$ and writing $v := u_N$ lead to

$$\begin{aligned}
&\{\min_{|u|=n} \overline{V}(u) > bn^{1/3}\} \\
\supset\ &\{\forall v \in \mathcal{T}_N, \overline{V}(v) > b_1 n^{1/3}, \forall u \in \mathcal{T}_{n-N}^v, \overline{V}^v(u) > b_2 n^{1/3}\} \\
\supset\ &\{\forall v \in \mathcal{T}_N, \overline{V}(v) > b_1 n^{1/3}, \#\mathcal{T} = m^N\} \cap \bigcap_{v \in \mathcal{T}_N} \{\forall u \in \mathcal{T}_{n-N}^v, \overline{V}^v(u) > b_2 n^{1/3}\}.
\end{aligned}$$

Using the branching property, we see that conditional on the history of the branching random walk up to generation N, the events $\left\{\min_{u \in \mathcal{T}_{n-N}^v} \overline{V}^v(u) > b_2 n^{1/3}\right\}$ are independent and have the same probability as $\left\{\min_{u \in \mathcal{T}_{n-N}} \overline{V}(u) > b_2 n^{1/3}\right\}$. Considering the probabilities of the events in the last display yields (2.2), since $\#\mathcal{T}_N \geq m^N$.

We turn to the proof of (2.3). As previously, we deduce from (2.4) that

$$\{\overline{V}(u) > bn^{1/3}\} \subset \{\overline{V}(u_N) > b_1 n^{1/3}\} \cup \{\overline{V}^{u_N}(u) > b_2 n^{1/3}\}.$$

Hence

$$\left\{\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right\} \subset \left\{\forall u \in \mathcal{T}_N, \left(\overline{V}(u) \le b_1 n^{1/3} \Rightarrow \min_{v \in \mathcal{T}^u_{n-N}} \overline{V}^u(v) > b_2 n^{1/3}\right)\right\}.$$

Conditional on the complementary of $L^\beta_N(b_1 n^{1/3})$, there are at least βm^N individuals –u^i, $1 \le i \le \lceil \beta m^N \rceil$ say– in generation N such that $\overline{V}(u^i) \le b_1 n^{1/3}$, $i = 1, \ldots, \lceil \beta m^N \rceil$, and the random variables $\min_{v \in \mathcal{T}^{u^i}_{n-N}} \overline{V}^{u^i}(v)$ are independent copies of $\min_{u \in \mathcal{T}_{n-N}} \overline{V}(u)$.
Equation (2.3) follows. □

3 The case of displacements bounded from above

We introduce $M =: \text{ess sup} X = \inf\{x \in \mathbb{R}, \mathbb{P}(X > x) = 0\}$. Our hypothesis imply $M > 0$.

Theorem 3.1. *If $m \ge 2$, $M < +\infty$ and $b > b_c$, then*

$$\frac{1}{n^{1/3}} \log\left(-\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right)\right) \to m^{\frac{(b-b_c)}{M}}.$$

Proof. We first prove the lower bound. Let $\varepsilon > 0$ and $N \in \mathbb{N}$. We consider the event E^ε_N that any $u \in \mathcal{T}$ such that $|u| < N$ has exactly m children, and that their displacements are all greater than $M - \varepsilon$. By construction,

$$E^\varepsilon_N \subset \left\{\min_{|u|=N} \overline{V}(u) > N(M-\varepsilon)\right\}.$$

By using independence, we obtain

$$\mathbb{P}\left(E^\varepsilon_N\right) = p_m^{\frac{m^N-1}{m-1}} \mathbb{P}(X > M-\varepsilon)^{\frac{m(m^N-1)}{m-1}}.$$

We set $b_1 := b - b_c + \varepsilon$, $b_2 := b_c - \varepsilon$ and $N := \left\lceil \frac{b_1 n^{1/3}}{M-\varepsilon} \right\rceil$. Last equation gives :

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log\left(-\log \mathbb{P}\left(\min_{|u|=N} \overline{V}(u) > b_1 n^{1/3}\right)\right) \le \frac{(b-b_c+\varepsilon)\log m}{M-\varepsilon}.$$

It remains to control the second factor in the right-hand side of (2.2). Since $N = o(n)$ and $b_2 = b_c - \varepsilon < b_c$, we already know that

$$\mathbb{P}(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}) \to 1 \text{ as } n \to \infty.$$

Hence, for n greater that some n_0,

$$\mathbb{P}\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right) > \frac{1}{2}.$$

This gives :

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log\left(-\log \mathbb{P}\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right)^{m^N}\right) \leq \frac{(b - b_c + \varepsilon)\log m}{M - \varepsilon}.$$

We have proved that both factors in the right-hand side of (2.2) decay at the same rate. Let $\varepsilon \to 0$ and the lower bound of Theorem 3.1 follows.

We turn to the proof of the upper bound. Our argument relies on (2.3) with $\beta = 1$. Let $\varepsilon > 0$. We set $b_1 := b - b_c - \varepsilon$ and $b_2 := b_c + \varepsilon$. For any $n \geq 1$, we choose $N := \left\lceil \frac{b_1 n^{1/3}}{M} \right\rceil - 1$. By construction,

$$\forall n \geq 1, \mathbb{P}\left(\exists u \in \mathcal{T}_N, \overline{V}(u) > b_1 n^{1/3}\right) = 0.$$

Since $b_1 < b_c$, we have, for n greater than some n_1,

$$\mathbb{P}\left(\min_{|u|=N} \overline{V}(u) > b_1 n^{1/3}\right) < \frac{1}{2}.$$

This gives :

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log\left(-\log \mathbb{P}\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right)^{m^N}\right) \geq \frac{(b - b_c - \varepsilon)\log m}{M}.$$

Letting $\epsilon \to 0$ yields the result. $\square$

4 The Gaussian-type case

The result in the case $M < +\infty$ says that $\mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right)$ has a double exponential decay when $b > b_c$; but the constant vanishes as $M \to \infty$, so that the decay is expected to be much slower when $M = +\infty$. Actually, the rate depends on the right tail of the distribution of X.

For any $\rho \in \mathbb{R}$, we denote by R_ρ the class of regularly varying functions of index ρ, in other words the functions f such that

$$\frac{f(\lambda x)}{f(x)} \to \lambda^\rho \text{ as } x \to +\infty.$$

We can now state our main result, which handles in particular the Gaussian case (with $\kappa = 2$) :

Theorem 4.1. *Assume that $m \geq 2$, and that $x \mapsto -\log \mathbb{P}(X > x)$ is regularly varying of index κ, where $1 < \kappa < +\infty$. Then, for any $b > b_c$, as $n \to \infty$,*

$$\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right) \sim \log \mathbb{P}\left(X > (b-b_c)n^{1/3}\right)\left(m^{\frac{1}{\kappa-1}} - 1\right)^{\kappa-1}.$$

Proof of the lower bound in Theorem 4.1. Set $N \in \mathbb{N}$, $N \geq 1$ (this time N is fixed and so does not depend on n). For $i = 1, \ldots, N$, define

$$a_i := m^{\frac{-i}{\kappa-1}}, \quad \lambda_i := \frac{a_i}{\sum_{i=1}^N a_i}.$$

Thanks to the branching property and the hypothesis $-\log \mathbb{P}(X > \cdot) \in R_\kappa$, we compute

$$\begin{aligned}
\mathbb{P}\left(\min_{|u|=N} \overline{V}(u) > b_1 n^{1/3}\right) &\geq \mathbb{P}\left(\#\mathcal{T}_N = m^N, \forall 1 \leq i \leq N, \forall u \in \mathcal{T}_i, X_u > \lambda_i b_1 n^{1/3}\right) \\
&\geq \mathbb{P}(\#\mathcal{T}_N = m^N) \prod_{i=1}^N \mathbb{P}\left(\forall u \in \mathcal{T}_i, X_u > \lambda_i b_1 n^{1/3} \,\middle|\, \#\mathcal{T}_N = m^N\right) \\
&\geq \mathbb{P}(\#\mathcal{T}_N = m^N) \prod_{i=1}^N \mathbb{P}\left(X > \lambda_i b_1 n^{1/3}\right)^{m^i} \\
&\geq \mathbb{P}(\#\mathcal{T}_N = m^N) \prod_{i=1}^N \mathbb{P}\left(X > b_1 n^{1/3}\right)^{(1+o(1))\lambda_i^\kappa m^i} \\
&\geq p_m^{\frac{m^N-1}{m-1}} \mathbb{P}\left(X > b_1 n^{1/3}\right)^{(1+o(1))\sum_{i=1}^N \lambda_i^\kappa m^i}. \qquad (4.1)
\end{aligned}$$

By our choice of λ_i and a_i, the exponent in the right-hand side becomes

$$\sum_{i=1}^N \lambda_i^\kappa m^i = \frac{\sum_{i=1}^N a_i^\kappa m^i}{\left(\sum_{i=1}^N a_i\right)^\kappa} = \frac{\sum_{i=1}^N m^{\left(\frac{-\kappa i}{\kappa-1}+i\right)}}{\left(\sum_{i=1}^N m^{\frac{-i}{\kappa-1}}\right)^\kappa} = \frac{1}{\left(\sum_{i=1}^N m^{\frac{-i}{\kappa-1}}\right)^{\kappa-1}} = \left(\frac{m^{\frac{1}{\kappa-1}} - 1}{1 - m^{\frac{-N}{\kappa-1}}}\right)^{\kappa-1}.$$

Hence we can rewrite (4.1) in the following way :

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\min_{|u|=N} \overline{V}(u) > b_1 n^{1/3}\right)}{\log \mathbb{P}\left(X > b_1 n^{1/3}\right)} \geq \left(\frac{m^{\frac{1}{\kappa-1}} - 1}{1 - m^{\frac{-N}{\kappa-1}}}\right)^{\kappa-1}.$$

This is a lower bound for the first factor of the right-hand side of (2.2). Take $b_1 > b - b_c$, then $b_2 > b_c$ and the second factor converges to 1 as n goes to infinity. Thus, combining (2.2) and the estimates above gives :

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right)}{\log \mathbb{P}\left(X > b_1 n^{1/3}\right)} \geq \left(\frac{m^{\frac{1}{\kappa-1}} - 1}{1 - m^{\frac{-N}{\kappa-1}}}\right)^{\kappa-1}.$$

Let $b_1 \to b - b_c$ and finally $N \to \infty$ and the lower bound is proved. □

Proving the upper bound requires several steps. We apply (2.3) twice : the first time (with large N) in order to obtain a rough estimate (Proposition 4.2). Then we apply (2.3) again and the previous estimate allows us to have a better control on the second term with N small enough so that we can control the first term (as stated in Proposition 4.3).

Proposition 4.2. *Assume $b > b_c$. Then the sequence*

$$n \mapsto -\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right)$$

is bounded from below by some regularly varying sequence of index κ.

Proof. We choose $0 < b_1 < b - b_c$, $b_2 = b - b_1 > b_c$ and

$$N = N(n) := \left\lfloor \kappa \frac{\log n}{\log m} \right\rfloor.$$

We have

$$L_N^1\left(b_1 n^{1/3}\right) \subset \left\{\exists u \in \mathcal{T}_n, \overline{V}(u) > b_1 n^{1/3}\right\} \subset \left\{\exists 1 \leq i \leq N, \exists u \in \mathcal{T}_i, X_{u_i} > \frac{b_1 n^{1/3}}{N}\right\}.$$

Hence

$$-\log \mathbb{P}\left(\max_{|u|=N} \overline{V}(u) > b_1 n^{1/3}\right) \geq -\log\left(m^N \frac{m}{m-1} \mathbb{P}\left(X > \frac{b_1 n^{1/3}}{N}\right)\right).$$

Since $n \mapsto N(n)$ is slowly varying, the right-hand side of the last inequality is regularly varying of index κ.

Since $b_2 > b_c$ and $N = o(n)$, we have, for any n large enough,

$$p\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right) \leq p\left(\min_{|u|=n-N} \overline{V}(u) > \frac{b_2 + b_c}{2}(n-N)^{1/3}\right) \leq \frac{1}{2}.$$

By our choice of N, this gives that

$$-\log \mathbb{P}\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right) \geq m^N \log 2 \sim n^\kappa \log 2.$$

Proposition 4.2 is a consequence of (2.3) with $\beta = 1$ and the estimates above. □

Proposition 4.3. *Let $b_1 > 0$ and $\gamma \in (0, \frac{\kappa-1}{3})$, $\gamma = \frac{\kappa-1}{4}$ say. Assume $n \mapsto N = N(n)$ is such that*

$$\inf_{n\geq 1} n^{\gamma} m^{-N} > 0 \text{ and } \lim_{n\to\infty} N = \infty.$$

Then

$$\lim_{\beta\to 0} \liminf_{n\to\infty} \frac{\log \mathbb{P}\left(L_N^{\beta}\left(b_1 n^{1/3}\right)\right)}{\log \mathbb{P}\left(X > b_1 n^{1/3}\right)} \geq \left(m^{\frac{1}{\kappa-1}} - 1\right)^{\kappa-1}.$$

The proof of Proposition 4.3 is postponed to the end of this section, after the proof of the upper bound in Theorem 4.1 (from Propositions 4.2 and 4.3) and the statement and the proof of preliminary results.

Proof of the upper bound in Theorem 4.1. We choose $0 < b_1 < b - b_c$, $b_2 = b - b_1 > b_c$ and set :

$$\forall n \geq 1, N = N(n) := \left\lfloor \varepsilon \frac{\log n}{\log m} \right\rfloor$$

where $\varepsilon > 0$ is small enough for the hypothesis of Proposition 4.3 to hold. Then we have

$$\lim_{\beta\to 0} \liminf_{n\to\infty} \frac{\log \mathbb{P}\left(L_N^{\beta}\left(b_1 n^{1/3}\right)\right)}{\log \mathbb{P}\left(X > b_1 n^{1/3}\right)} \geq \left(m^{\frac{1}{\kappa-1}} - 1\right)^{\kappa-1}.$$

In light of Proposition 4.2, we see that the sequence :

$$n \mapsto -\log\left(\mathbb{P}\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right)^{\beta m^N}\right)$$

is bounded from below by a regularly varying sequence of index $\kappa + \beta\varepsilon$.

Thus, the second term in the right-hand side of (2.3) is negligible and this inequality yields

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > b n^{1/3}\right)}{\log \mathbb{P}\left(X > b_1 n^{1/3}\right)} \geq \left(m^{\frac{1}{\kappa-1}} - 1\right)^{\kappa-1}.$$

Since the function $x \mapsto -\log \mathbb{P}(X > x)$ is regularly varying of index κ, the last display becomes

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > b n^{1/3}\right)}{\log \mathbb{P}\left(X > (b - b_c) n^{1/3}\right)} \geq \left(\frac{b_1}{b - b_c}\right)^{\kappa} \left(m^{\frac{1}{\kappa-1}} - 1\right)^{\kappa-1}.$$

We conclude by letting $b_1 \to b - b_c$. $\square$

It remains to prove Proposition 4.3.

For any $u \in \mathcal{T}$, we consider $X_u^* := X_u^+$ the positive part of X_u and, accordingly,

$$V^*(u) := \sum_{v \le u} X_v^*.$$

We could define, in the same fashion,

$$\overline{V}^*(u) := \max_{v \le u} X_v^*,$$

but we would have, since the X_u^* are nonnegative, $\overline{V}^*(u) = V^*(u)$.
Obviously we have, for any $u \in \mathcal{T}$, $V^*(u) \ge V(u)$ and thus $V^*(u) \ge \overline{V}(u)$.
We define

$$S_N := \sum_{u \in \widetilde{\mathcal{T}}_N} V^*(u) m^{-N}.$$

Since each X_u^+ is counted once for each descendant of u in $\widetilde{\mathcal{T}}_N$, that is to say $m^{N-|u|}$ times, we have

$$S_N = \sum_{k=1}^{N} m^{-k} \sum_{u \in \widetilde{\mathcal{T}}_k} X_u^+. \tag{4.2}$$

The relevance of these definitions is made clear by the following lemma :

Lemma 4.4. *For any $x \in \mathbb{R}$, $\beta \in (0,1]$ and $N \in \mathbb{N}$,*

$$\mathbb{P}\left(L_N^\beta(x)\right) \le \mathbb{P}\left(S_N > (1-\beta)x\right).$$

Proof. The lemma is a consequence of the inclusion :

$$\begin{aligned}
L_N^\beta(x) &\subset \left\{\#\left\{u \in \widetilde{\mathcal{T}}_N : \overline{V}(u) \le x\right\} < \beta m^N\right\} \\
&\subset \left\{\#\left\{u \in \widetilde{\mathcal{T}}_N : V^*(u) \le x\right\} < \beta m^N\right\} \\
&\subset \left\{\#\left\{u \in \widetilde{\mathcal{T}}_N : V^*(u) > x\right\} \ge (1-\beta) m^N\right\} \\
&\subset \left\{S_N > (1-\beta)x\right\}.
\end{aligned}$$

□

If, for some $\rho > 0$ and $a > 0$, $f \in R_\rho$ is defined and locally bounded on $[a, +\infty)$, then we can define the generalized inverse

$$\overleftarrow{f}(x) := \inf\{y \ge a : f(y) > x\}.$$

We recall some usefull properties (and refer to [17] for the proofs) : $\overleftarrow{f} \in R_{\rho^{-1}}$ and $\overleftarrow{\overleftarrow{f}}(x) \sim f(x)$ as $x \to +\infty$. Here we deal with functions f that are increasing and right-continuous, which implies $\overleftarrow{\overleftarrow{f}}(x) = f(x)$ for any x large enough.

Theorem 4.5 (Kasahara's Tauberian theorem (1978))**.** *Let Y be a nonnegative random variable such that, for all $\lambda > 0$,*

$$\Lambda_Y(\lambda) := \mathbb{E}\left[e^{\lambda Y}\right] < +\infty$$

If $0 < \alpha < 1$, $\Phi \in R_\alpha$, put $\Psi(\lambda) := \frac{\lambda}{\Phi(\lambda)} \in R_{1-\alpha}$. Then, for $B > 0$,

$$-\log \mathbb{P}(Y > x) \sim B\overleftarrow{\Phi}(x) \text{ as } x \to +\infty$$

if and only if

$$\Lambda_Y(\lambda) \sim (1-\alpha)\left(\frac{\alpha}{B}\right)^{\frac{\alpha}{1-\alpha}} \overleftarrow{\Psi}(\lambda) \text{ as } \lambda \to +\infty.$$

Again we refer to [17] (Theorem 4.12.7) for the proof.

We state a last lemma before the proof of Proposition 4.3 :

Lemma 4.6. *Let $\rho > 0$ and $\delta > 0$ be such that $\delta < \rho$. Assume that $f \in R_\rho$ and that $g : x \mapsto g(x) := x^{-\delta} f(x)$ is bounded on each interval $(0, a]$ ($a > 0$). Then*

$$\forall \varepsilon > 0, \exists x_0 > 0, \forall x > x_0, \forall 0 < \lambda \leq 1, \left|\frac{f(\lambda x)}{f(x)} - \lambda^\rho\right| \leq \varepsilon \lambda^\delta.$$

Proof. Clearly, $g \in R_{\rho-\delta}$. Thus, by the Uniform Convergence Theorem (see [17], Theorem 1.5.2), we have,

$$\forall \varepsilon > 0, \exists x_0 > 0, \forall x > x_0, \forall 0 < \lambda \leq 1, \left|\frac{g(\lambda x)}{g(x)} - \lambda^{\rho-\delta}\right| = \left|\frac{f(\lambda x)}{\lambda^\delta f(x)} - \lambda^{\rho-\delta}\right| \leq \varepsilon.$$

We conclude by multiplying both sides of the inequality by λ^δ. $\square$

Proof of Proposition 4.3. From now on, we set

$$\alpha := \frac{1}{\kappa}, \quad \Phi := \overleftarrow{-\log \mathbb{P}\left(X^+ > \cdot\right)} \text{ and } \Psi(\lambda) := \frac{\lambda}{\Phi(\lambda)} \in R_{1-\alpha}.$$

By taking $B = 1$ and $Y = X^+$, Theorem 4.5 gives

$$\Lambda_{X^+}(t) \sim (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} \overleftarrow{\Psi}(t) \text{ as } \to +\infty. \tag{4.3}$$

By using (4.2) and the independence of the X_u, $u \in \mathcal{T}$, we can compute the moment generating function of S_N :

$$\forall t > 0, \Lambda_{S_N}(t) := \log \mathbb{E}\left[\mathrm{e}^{tS_N}\right] = \sum_{k=1}^{N} \sum_{u \in \widetilde{\mathcal{T}}_k} \log \mathbb{E}\left[\mathrm{e}^{tX_u^+ m^{-k}}\right] = \sum_{k=1}^{N} m^k \Lambda_{X^+}\left(tm^{-k}\right).$$

Let $a > 0$ and $\delta \in (1, (1-\alpha)^{-1})$. We may alter Λ_{X^+} on $(0, a)$ and obtain $\widetilde{\Lambda_{X^+}}$ such that $t \mapsto t^{-\delta}\widetilde{\Lambda_{X^+}}(t)$ is locally bounded. Then, since $\Lambda_{X^+} \in R_{(1-\alpha)^{-1}}$, we may apply Lemma 4.6 to $\widetilde{\Lambda_{X^+}}$. Because $\widetilde{\Lambda_{X^+}}$ and Λ_{X^+} are equal on $[a, +\infty)$, we obtain :

$$M_{a,\delta}(t) := \sup_{\frac{a}{t} \leq \lambda \leq 1} \lambda^{-\delta} \left| \frac{\Lambda_{X^+}(\lambda t)}{\Lambda_{X^+}(t)} - \lambda^{\frac{1}{1-\alpha}} \right| \to 0 \text{ as } t \to +\infty.$$

By taking $\lambda = m^{-1}, m^{-2}, \ldots, m^{-N}$, in the last display we deduce from equation (4) that, for any t and N such that $t \geq am^N$,

$$\frac{\Lambda_{S_N}(t)}{\Lambda_{X^+}(t)} \leq \sum_{k=1}^{N} m^{k\left(1-\frac{1}{1-\alpha}\right)} + m^{k(1-\delta)} M_{a,\delta}(t).$$

If $N = N(t)$ is such that $N \to \infty$ as $t \to +\infty$ and for all t large enough, $t \geq am^N$, then the right-hand side of the inequality is equivalent, as $t \to +\infty$, to :

$$\sum_{k=1}^{\infty} m^{-k\frac{\alpha}{1-\alpha}} = \frac{1}{m^{\frac{\alpha}{1-\alpha}} - 1}.$$

By Markov's inequality, for any $x > 0$, for any $t \geq 0$, we have

$$\mathbb{P}\left(S_N > x\right) \leq \Lambda_{S_N}(t)\mathrm{e}^{-tx}.$$

As a consequence, taking $x = \Phi(y)$ and $t = \Psi(\lambda y)$ for any $y > 0$ and $\lambda > 0$ yields

$$\log \mathbb{P}\left(S_N > \Phi(y)\right) \leq \Lambda_{S_N}(\Psi(\lambda y)) - \Psi(\lambda y)\Phi(y). \tag{4.4}$$

Since $\Psi(y)\Phi(y) = y$ and $\Psi \in R_{1-\alpha}$, we have, as $y \to \infty$, $\Psi(\lambda y)\Phi(y) \sim y\lambda^{1-\alpha}$. Uniformly in λ and N such that $\Psi(\lambda y)m^{-N} \geq a$, we have, as $y \to \infty$,

$$\Lambda_{S_N}(\Psi(\lambda y)) \leq \Lambda_{X^+}(\Psi(\lambda y))\frac{1 + o(1)}{m^{\frac{\alpha}{1-\alpha}} - 1}.$$

By (4.3), we have

$$\Lambda_{X^+}\left(\Psi(y)\right) \sim (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} y \text{ as } y \to +\infty.$$

Then, by combining, (4.4) becomes

$$\log \mathbb{P}\left(S_N > \Phi(y)\right) \leq \frac{1+o(1)}{m^{\frac{\alpha}{1-\alpha}}-1}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\lambda y - y\lambda^{1-\alpha}(1+o(1)).$$

We follow the proof of Theorem 4.5 in [17] and take, frow now on,

$$B := \left(m^{\frac{\alpha}{1-\alpha}}-1\right)^{\frac{1-\alpha}{\alpha}} = \left(m^{\frac{1}{\kappa-1}}-1\right)^{\kappa-1}.$$

The function

$$\lambda \mapsto \left(\frac{\alpha}{B}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha)\lambda - \lambda^{1-\alpha}$$

is strictly convex on $(0,+\infty)$, attaining its unique minimum $-B$ at

$$\lambda_0 = \left(\frac{B}{\alpha}\right)^{\frac{1}{1-\alpha}}.$$

Let $\beta \in (0,1]$. Take $y = \overleftarrow{\Phi}((1-\beta)b_1 n^{1/3}) = -\log \mathbb{P}(X > b_1(1-\beta)n^{1/3})$, this yields

$$\log \mathbb{P}\left(S_N > (1-\beta)b_1 n^{1/3}\right) \leq -By(1+o(1)) = B(1+o(1))\log \mathbb{P}(X > (1-\beta)b_1 n^{1/3})$$

on condition that $\Psi(\lambda_0 \overleftarrow{\Phi}((1-\beta)b_1 n^{1/3}))m^{-N} \geq a$ for any n large enough.

Since the sequence $n \mapsto \Psi(\lambda_0 \overleftarrow{\Phi}(n^{1/3}))$ is regularly varying of index $\frac{1-\alpha}{3\alpha} = \frac{\kappa-1}{3} > \gamma$, our condition on N is a consequence of the assumption of Proposition 4.3, and letting $\beta \to 0$ completes the proof. □

5 The exponential case

Theorem 5.1. *Let $b > b_c$. Let $t_0 := \sup\{t \in \mathbb{R} : \Lambda_X(t) < +\infty\} \in (0,+\infty]$. Then*

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right)}{(b-b_c)n^{1/3}} \leq -t_0 m. \tag{5.1}$$

If $x \mapsto \mathbb{P}(X > x)$ is regularly varying (of index 1, it is also true if the index is greater but in this case the conclusion of Theorem 4.1 is stronger), then

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right)}{\log \mathbb{P}\left(X > (b-b_c)n^{1/3}\right)} \leq m \tag{5.2}$$

Remark 5.2. Notice that

$$t_0 = -\limsup_{x\to+\infty} \frac{\log \mathbb{P}(X > x)}{x}.$$

As a consequence when this lim sup is a limit and is finite, the bounds in the theorem above agree.

Proof of Theorem 5.1. Let $b_1 > b - b_c$ and $b_2 = b - b_1 < b_c$. Set $N = 1$. Then (2.2) gives

$$\mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right) \geq p_m \mathbb{P}\left(X > b_1 n^{1/3}\right)^m \mathbb{P}\left(\min_{|u|=n-1} \overline{V}(u) > b_2 n^{1/3}\right)^m$$

Since $b_2 < b_c$ we deduce from (1.1) :

$$\mathbb{P}\left(\min_{|u|=n-1} \overline{V}(u) > b_2 n^{1/3}\right) \to 1.$$

Hence

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right)}{\log \mathbb{P}\left(X > b_1 n^{1/3}\right)} \leq m$$

We obtain (5.2) by letting $b_1 \to b - b_c$.

We turn to the proof of (5.1). Let $\beta \in (0,1]$, $b_1 < b - b_c$, $b_2 = b - n_1 > b_c$ and $t \in (0, t_0)$. Then $\Lambda_X(t) < +\infty$. Let $N \in \mathbb{N}$. We have, like in the proof of Theorem 4.1,

$$\mathbb{P}\left(L_N^\beta(b_1 n^{1/3})\right) \leq \mathbb{P}\left(S_N > (1-\beta) b_1 n^{1/3}\right) \leq \exp\left(\Lambda_{S_N}(t) - t(1-\beta) b_1 n^{1/3}\right).$$

For any $\lambda \in (0,1)$, by convexity of Λ_{X^+},

$$\Lambda_{X^+}(\lambda t) \leq \lambda \Lambda_{X^+}(t) + (1-\lambda)\Lambda_{X^+}(\lambda 0) = \lambda \Lambda_{X^+}(t).$$

Hence

$$\Lambda_{S_N}(t) = \sum_{k=1}^{N} m^k \Lambda_{X^+}\left(tm^{-k}\right) \leq N \Lambda_{X^+}(t).$$

Let $\varepsilon_1 > 0$. Set $N := \lceil \varepsilon_1 n^{1/3} \rceil$. We obtain, by Lemma 4.4 and Markov's inequality,

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(L_N^\beta(b_1 n^{1/3})\right)}{n^{1/3}} \leq \varepsilon_1 \Lambda_{X^+}(t) - t(1-\beta) b_1.$$

Let $\varepsilon_2 > 0$. Since $b_2 > b_c$, (1.1) yields that for n large enough,

$$\mathbb{P}\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right) \leq \varepsilon_2.$$

We combine the estimates above with (2.3), which gives

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right)}{n^{1/3}} \leq \max\left\{\varepsilon_1 \Lambda_{X^+}(t) - t(1-\beta) b_1, \varepsilon_1 \beta \log \varepsilon_2\right\}.$$

We conclude by letting $\varepsilon_2 \to 0$, then $\varepsilon_1 \to 0$, and finally $b_1 \to b - b_c$, $\beta \to 0$ and $t \to t_0$. $\square$

6 The case $m = 1$

Theorem 6.1. *If $m = 1$, then there are constants $0 < c_2 \leq c_1 < +\infty$ such that for any $b > b_c$,*

$$-c_1 \leq \liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right)}{(b-b_c)n^{1/3}}; \tag{6.1}$$

$$-c_2 \geq \limsup_{n\to\infty} \frac{\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right)}{(b-b_c)n^{1/3}}. \tag{6.2}$$

Proof of (6.1). Let $b > b_c$, $b_1 > b - b_c$ and $b_2 = b - b_1 < b_c$. Let $A > 0$ be such that $\mathbb{P}(X > A) > 0$.

Take $N := \left\lceil \frac{b_1 n^{1/3}}{A} \right\rceil$ and proceed like in the proof of the lower bound of Theorem 3.1 to obtain

$$\mathbb{P}\left(\#\mathcal{T}_N = 1; \forall i \geq N, \forall u \in \mathcal{T}_i, X_u > A\right) \geq \left(p_1 \mathbb{P}(X > A)\right)^N.$$

By (1.1), we have

$$\lim_{n\to\infty} \mathbb{P}\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right) = 1.$$

Combining the two estimates above with (2.2) yields

$$-c_1 \leq \liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right)}{b_1 n^{1/3}}$$

where

$$c_1 := \frac{-\log\left(p_1 \mathbb{P}(X > A)\right)}{A}.$$

We conclude the proof of the lower bound by letting $b_1 \to b - b_c$. $\square$

Before proving the upper bound (6.2), we state a simple result for Galton-Watson processes :

Lemma 6.2. *Let $\mathcal{T}$ be a supercritical Galton-Watson process with extinction probability $q < 1$. Then there exists $p \in (0,1)$ and $s \in (0,1)$ such that*

$$\mathbb{E}\left[s^{\#\mathcal{T}_n}\right] \leq q + p^n.$$

Proof. Define the moment generating functions $f_n : s \in [0,1] \mapsto \mathbb{E}\left[s^{\#\mathcal{T}_n}\right]$ for all $n \geq 1$. By the branching property, $f_{n+1} = \sum_{i=0}^{\infty} p_i f_n(\cdot)^i = f_1 \circ f_n$, hence $f_n = f_1^{\circ n}$. Since the process is supercritical, $f_1'(1) = \sum_i ip_i > 1$. Besides, f_1 is increasing and convex, with $f_1(1) = 1$, hence $0 \le f_1'(q) < 1$. Let $p \in (f_1'(q), 1)$. Then for some $s_0 > q$, we have

$$\forall s \in [q, s_0], f_1(s) - q \le p(s-q).$$

Then for any $s \in [q, s_0]$, we obtain by induction $f_n(s) - q \le p^n(s-q)$. $\square$

Proof of (6.2). Let $b > b_c$, $b_1 < b - b_c$ and $b_2 = b - b_1 > b_c$. Let $\beta \in (0,1)$.
The argument we used in the preceding sections requires some changes here. Let $N \le n$ and $A_N \ge 1$ be integers. We can, conditional on $\{\#\mathcal{T}_N \ge A_N\}$, choose A_N distinct individuals $u^1, u^2, \ldots, u^{A_N}$ in $\mathcal{T}_N$ in a measurable way (with respect to the σ-field $\mathcal{F}_N^{\text{G-W}}$ generated by the Galton-Watson process up to generation N), for example by taking the A_N first ones in lexicographic ordering. Hence we can define a random variable $S_N^{A_N}$ such that

$$S_N := \begin{cases} \frac{1}{A_N}\mathbb{E}\left[\sum_{i=1}^{A_N} V^*(u^i)\right] = \sum_{k=1}^N \sum_{u\in\mathcal{T}_k} \lambda_u(\mathcal{T}) X_u^+ \text{ on } \{\#\mathcal{T}_N \ge A_N\}, \\ 0 \text{ if } \{\#\mathcal{T}_N < A_N\}. \end{cases}$$

where, for any potential individual u such that $|u| \le N$,

$$\lambda_u(\mathcal{T}) := A_N^{-1}\#\{i \le A_N : u^i \ge u\} \in [0,1]$$

is a random variable measurable with respect to $\mathcal{F}_N^{\text{G-W}}$ and thus independent of the X_u. Conditional on $\{\#\mathcal{T}_N \ge A_N\}$, we have

$$\forall 1 \le k \le N, \quad \sum_{u\in\mathcal{T}_N} \lambda_u(\mathcal{T}) = 1.$$

Therefore the convexity of Λ_{X^+} gives the inequality :

$$\begin{aligned} \mathbb{E}\left[e^{tS_N}\mathbb{I}_{\{\#\mathcal{T}_N \ge A_N\}}\right] &= \mathbb{E}\left[\mathbb{I}_{\{\#\mathcal{T}_N\ge A_N\}}\mathbb{E}\left[e^{tS_N} \,\middle|\, \mathcal{F}_N^{\text{G-W}}\right]\right] \\ &= \mathbb{E}\left[\mathbb{I}_{\{\#\mathcal{T}_N\ge A_N\}} \exp\left(\sum_{k=1}^N \sum_{u\in\mathcal{T}_k} \Lambda_{X^+}(t\lambda_u(\mathcal{T}))\right)\right] \\ &\le \mathbb{E}\left[\mathbb{I}_{\{\#\mathcal{T}_N\ge A_N\}} \exp\left(N\Lambda_{X^+}(t)\right)\right] = \mathbb{P}\left(\#\mathcal{T}_N \ge A_N\right)\exp\left(N\Lambda_{X^+}(t)\right) \\ &\le \exp\left(N\Lambda_{X^+}(t)\right). \end{aligned} \tag{6.3}$$

The analogous of (2.3) in this case, with A_N playing the role of m^N, is

$$\begin{aligned} \mathbb{P}\left(\min_{|u|=n} \overline{V}(u) > bn^{1/3}\right) &\le \mathbb{P}\left(\#\{u \in \mathcal{T}_N : \overline{V}(u) > b_1 n^{1/3}\} > (1-\beta)A_N\right) \\ &\quad + \mathbb{P}\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right)^{\beta A_N}. \end{aligned}$$

The first term of the right-hand side splits into two terms :

$$\begin{aligned}\mathbb{P}\left(\#\{u \in \mathcal{T}_N : \overline{V}(u) > b_1 n^{1/3}\} > (1-\beta)A_N\right) &\leq \mathbb{P}\left(\#\mathcal{T}_N < A_N\right) \\ &\quad + \mathbb{P}\left(\#\mathcal{T}_N \geq A_N, S_N > (1-\beta)b_1 n^{1/3}\right).\end{aligned}$$

Markov's inequality and (6.3) give us for the last term

$$\begin{aligned}\mathbb{P}\left(\#\mathcal{T}_N \geq A_N, S_N > (1-\beta)b_1 n^{1/3}\right) &\leq \mathbb{E}\left[\mathrm{e}^{t\left(S_N-(1-\beta)b_1 n^{1/3}\right)}\mathbb{1}_{\{\#\mathcal{T}_N \geq A_N\}}\right] \\ &\leq \exp\left(N\Lambda_{X^+}(t) - t(1-\beta)b_1 n^{1/3}\right).\end{aligned}$$

Let $\varepsilon_1 > 0$ be such that $\varepsilon_1 \Lambda_{X^+}(t) < t(1-\beta)$. We set, for all $n \geq 0$,

$$N := \lceil \varepsilon_1 b_1 n^{1/3} \rceil.$$

This gives

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\left(\#\mathcal{T}_N \geq A_N, S_N > (1-\beta)b_1 n^{1/3}\right) \leq -(1-\beta)t - \varepsilon_1 \Lambda_{X^+}(t))b_1.$$

For any $s \in (0,1)$, we have

$$\mathbb{P}\left(\#\mathcal{T}_N < A_n\right) \leq \mathbb{E}\left[s^{\#\mathcal{T}_N}\right] s^{-A_n}.$$

Let p and s like in Lemma 6.2. We obtain

$$\mathbb{P}\left(\#\mathcal{T}_N < A_n\right) \leq (q + p^N)s^{-A_n}.$$

Since $m = 1$, we have $q = 0$. Let $\varepsilon_2 > 0$ be such that $\varepsilon_1 \log p < \varepsilon_2 \log s$. We set, for all $n \geq 0$,

$$A_n = \lceil \varepsilon_2 b_1 n^{1/3} \rceil.$$

We obtain

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\left(\#\mathcal{T}_N < A_n\right) \leq (\varepsilon_1 \log p - \varepsilon_2 \log s)b_1.$$

Since $b_2 > b_c$, (1.1) shows that for any $\varepsilon_3 > 0$, for any n large enough,

$$\mathbb{P}\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right) < \varepsilon_3.$$

Then

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \left(\mathbb{P}\left(\min_{|u|=n-N} \overline{V}(u) > b_2 n^{1/3}\right)^{\beta A_n}\right) \leq b_1 \beta \varepsilon_2 \log \varepsilon_3.$$

Combining yields (6.2) with

$$c_2 = \min\{\beta\varepsilon_2 \log s - \varepsilon_1 \log p, (1-\beta)t - \varepsilon_1 \Lambda_{X^+}(t), -\beta\varepsilon_2 \log \varepsilon_3\}.$$

It remains to check that we can adjust the parameters in such a way that $c_2 > 0$. We remark that the good choices for ε_3 and ε_2 is to send them to 0 in this order, which gives the following constant :

$$c_2 = \min\{-\varepsilon_1 \log p, (1-\beta)t - \varepsilon_1 \Lambda_{X^+}(t)\}.$$

Since this value is clearly positive for ε_1 small enough, Theorem 6.1 is proved. □

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